

# Elastic Green's Functions for Anisotropic Solids

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## 1. Introduction

We briefly review the Green's function method for solution of the Christoffel equation and a computationally efficient method for calculating the Green's function for anisotropic solids. We describe the virtual-force method for satisfying the boundary conditions to account for discontinuities in the solids. The virtual-force method consists of applying a distribution of virtual forces just outside the domain of solution. The solution obtained by using the virtual-force distribution and the Green's function gives a solution of the homogeneous equation. The virtual-force distribution is then determined by imposing the prescribed boundary conditions. This method is similar to the image-charge method in electrostatics [1] and is the basis for the boundary-element method [2] for solving elastodynamic problems.

The Fourier representation of the Green's function is quite general and, subject to certain well-known conditions of integrability and convergence [1], can be used for most physical problems. In the case of elastodynamic Green's functions, the Fourier representation is CPU intensive and is not computationally efficient for anisotropic solids. We have developed a delta-function representation [3] that is particularly suitable for anisotropic solids. In this paper, we describe the delta-function representation for elastodynamic as well as elastostatic Green's functions for infinite solids and its application to bounded solids by using the virtual-force method.

## 2. Green's function method

We represent the space and time variables by  $\mathbf{x}$  and  $t$ , respectively. The Cartesian components of a vector will be denoted by indices  $i, j, k$ , etc., which assume the values 1, 2, or 3, corresponding to  $x, y$ , and  $z$  coordinates. Summation over repeated Roman indices will be implied unless stated otherwise.

The Christoffel equation for elastic equilibrium is written in the operator form as

$$L_{ij} u_j(\mathbf{x}, t) = F_i(\mathbf{x}, t), \quad (1)$$

where

$$L_{ij} = c_{ijkl} \partial^2 / \partial x_k \partial x_l - \delta_{ij} \rho \partial^2 / \partial t^2, \quad (2)$$

$c$  is the fourth-rank elastic-constant tensor,  $\rho$  is the density of the solid, and  $\mathbf{F}(\mathbf{x},t)$  is the applied force. The Green's function  $\mathbf{G}(\mathbf{x},\mathbf{x}';t,t')$  is the solution of the equation

$$\mathbf{L}_{ij} \mathbf{G}_{jk}(\mathbf{x},\mathbf{x}';t,t') = \delta_{ik} \delta(\mathbf{x}-\mathbf{x}') \delta(t-t'), \quad (3)$$

where  $\mathbf{x}'$  and  $t'$  are variables in the same space as  $\mathbf{x}$  and  $t$ , respectively;  $\delta_{ik}$  is the Kronecker's delta tensor that is 1 for  $i=j$ , and 0 otherwise; and  $\delta(\mathbf{x})$  is the Dirac delta function defined by the relation

$$w(a) = \int \delta(p-a) w(p) dp. \quad (4)$$

In eq (4),  $a$  is a constant on the real axis and  $w(p)$  is any arbitrary integrable function of  $p$ , and the integration is over the entire real axis. We have used the same symbol for the Dirac delta function and the Kronecker delta tensor since they can be identified by their arguments.

The particular solution of eq (1), which can be verified by applying the operator  $\mathbf{L}$  and using eqs (3) and (4), is given by

$$\mathbf{u}_p(\mathbf{x},t) = \int \mathbf{G}(\mathbf{x},\mathbf{x}';t,t') \mathbf{F}(\mathbf{x}',t') d\mathbf{x}' dt'. \quad (5)$$

The integration in eq (5) is over the entire space of  $\mathbf{x}$  and  $t$ . For an infinite solid with no boundary conditions prescribed over space, eq (5) gives the final solution. For infinite solids,  $\mathbf{G}$  depends on  $\mathbf{x}$  and  $\mathbf{x}'$  and  $t$  and  $t'$  only through their differences  $\mathbf{x}-\mathbf{x}'$  and  $t-t'$ , respectively. In such cases  $\mathbf{G}$  can be denoted by a single space and time variable as  $\mathbf{G}(\mathbf{x}-\mathbf{x}', t-t')$  or  $\mathbf{G}(\mathbf{x},t)$  that implies  $\mathbf{x}'=t'=0$ .

For solids with spatial discontinuities, such as bounded solids or those containing holes, crack, interfaces, etc., we need to satisfy some prescribed boundary conditions. Let  $S$  specify the space of the solid over which we need to solve eq (1) with boundary conditions prescribed over the surface of  $S$ . Equation (5) still gives the particular solution of eq (1) with the integration over  $\mathbf{x}'$  restricted to  $S$ . To obtain the homogeneous solution, we apply a distribution of virtual forces  $\mathbf{f}(\mathbf{x}_S,t)$  over the surface of  $S$ . The function  $\mathbf{f}(\mathbf{x},t)$  is 0 everywhere except for  $\mathbf{x} = \mathbf{x}_S$ , where  $\mathbf{x}_S$  lies over the surface of  $S$ . The homogeneous solution, as in eq (5), is given by

$$\mathbf{u}_H(\mathbf{x},t) = \int \mathbf{G}(\mathbf{x}-\mathbf{x}'; t-t') \mathbf{f}(\mathbf{x}',t') \delta(\mathbf{x}'-\mathbf{x}_S) d\mathbf{x}' dt'. \quad (6)$$

Since eq (6) gives the homogeneous solution for arbitrary  $\mathbf{f}(\mathbf{x}_S,t)$ , we determine this function by imposing the boundary conditions. This is the essence of the Green's function method. For solids with simple geometrical surfaces, we can determine  $\mathbf{f}(\mathbf{x}_S,t)$  analytically. For solids with complicated geometrical shapes,  $\mathbf{f}(\mathbf{x}_S,t)$  has to be determined numerically as is done in the boundary-element method. The solution of eq (1), that is the

displacement field  $\mathbf{u}(\mathbf{x},t)$  for an applied unit force, subject to all the prescribed boundary conditions, is the Green's function for the solid.

The above derivation shows how Green's function for any geometrical structure can be built up in stages or modules. For example if  $\mathbf{G}(\mathbf{x},t)$  is the free space Green's function for an infinite solid, then to obtain Green's function for a semi-infinite solid with one free surface, we apply a virtual force just outside the free surface and add a homogeneous solution to the solution for the infinite solid. We then determine the virtual force by applying the boundary condition at the free surface. The Green's function thus obtained will be the Green's function for the semi-infinite solid. If we want to add another free surface or any other discontinuity like a hole in the solid, we apply another virtual force at the new surface or the discontinuity. We add this homogeneous solution to that obtained by using the Green's function for the semi-infinite solid. We determine the new virtual force by imposing the additional boundary conditions at the discontinuity. Only the additional boundary condition needs to be satisfied since the semi-infinite Green's function will automatically satisfy the boundary condition at the first free surface.

In the linear case,  $\mathbf{f}(\mathbf{x}_s, t)$  will be proportional to  $\mathbf{F}(\mathbf{x},t)$ . Hence the boundary conditions will be satisfied for all  $\mathbf{F}(\mathbf{x},t)$ . The Green's function thus obtained will therefore be independent of  $\mathbf{F}(\mathbf{x},t)$ . The Green's function is a characteristic of the solid including discontinuities, if any, and does not need to be recalculated for a different applied load. It should be useful, therefore, that the Green's functions for typical geometrical shapes can be stored in a central computer and made available to other users. The idea of such a library of Green's functions has been recently suggested by Rizzo [4].

### 3. Integral representation for Green's functions

In general it is possible to use the three-dimensional (3D) Fourier integral representation for the Green's function as given below:

$$\mathbf{G}(\mathbf{x},t) = (2\pi)^{-4} \int \mathbf{G}_F(\mathbf{K}, \omega) \exp i(\mathbf{K} \cdot \mathbf{x} - \omega t) d\mathbf{K} d\omega \quad (7)$$

where  $i^2 = -1$ ,  $\mathbf{G}_F(\mathbf{K}, \omega)$  is the Fourier transform of the Green's function,  $\mathbf{K}$  is the wave vector, and  $\omega$  is the frequency; and the integration in eq (7) is over the entire space. From eqs (2) and (3)

$$\mathbf{G}_F(\mathbf{K}, \omega) = [\Lambda(\mathbf{K}) - I \rho \omega^2]^{-1}, \quad (8)$$

where

$$\Lambda_{ij}(\mathbf{K}) = c_{ijkl} K_k K_l. \quad (9)$$

The 3x3 matrix  $\Lambda(\mathbf{K})$  is the Christoffel matrix in Fourier space. It is the long-wavelength (low-K) limit of the Born-von Karman dynamical matrix [5]. Its eigenvalues  $\omega^2(\mathbf{K})$  are the

squares of the phonon frequencies and its eigenvectors are the polarization vectors of the corresponding phonons. Equation (7) along with eqs (8) and (9) can be used for calculating the Green's function. The Green's function thus calculated will not be causal. To ensure causality, we can introduce a small imaginary part in  $\omega$  [5] and take the limit as the imaginary part approaches 0. Alternatively, we can take the Laplace transform over time and use the Laplace inversion integral in eq (7).

For an isotropic solid, the matrix inversion in eq (8) and the integration in eq (7) can be done analytically. The Green's function can also be obtained analytically [6] from eq (7) for a line force, or a 2D approximation, since the component of  $\mathbf{K}$  in the direction of the line force is 0. For a general 3D anisotropic solid, eq (7) requires a 4D numerical integration– over three components of  $\mathbf{K}$ , and one frequency variable. In general  $\mathbf{G}_F(\mathbf{K}, \omega)$  has singularities (resonances) on the real axis. The integral in eq (7) is defined in the Cauchy sense. It involves evaluation of principal values that creates problems of numerical convergence. A numerical evaluation of the general 3D anisotropic Green's function using the Fourier representation is CPU intensive. Some shortcuts for evaluating the Green's function are available in the literature (see refs. [3] and [7] and other references quoted therein).

We have developed a delta-function representation of the Green's function [3] that is computationally convenient even for 3D anisotropic solids. In this representation we write

$$\mathbf{G}(\mathbf{x},t) = (4\pi^3)^{-1} \int \mathbf{G}_q(\mathbf{q}) \delta^{(1)}(t-\mathbf{q}\cdot\mathbf{x}) H(t) d\mathbf{q}, \quad (10)$$

where

$$\mathbf{G}_q(\mathbf{q}) = \text{Lim}_{\epsilon \rightarrow +0} \text{Im} [\Lambda(\mathbf{q}) - (1-t\epsilon) \mathbf{I}]^{-1}, \quad (11)$$

$$\Lambda_{ij}(\mathbf{q}) = c_{ijkl} q_k q_l, \quad (12)$$

$H(t)$  is the Heaviside step function, being unity for  $t > 0$  and 0 for  $t < 0$ ,  $\delta^{(1)}$  is the first derivative of the delta function with respect to its argument, and  $\mathbf{q}$ , that has the dimensions of inverse velocity, is a vector in slowness space. We identify  $\Lambda(\mathbf{q})$  as the Christoffel matrix and  $\mathbf{G}_q(\mathbf{q})$  as the Green's function in the slowness space. The delta function in eq (10) is a statement of the physical fact that a phonon of slowness value  $\mathbf{q}$  will reach the distance  $x$  in time  $t$ , then its velocity  $1/q$  must be  $x/t$ . The function  $\mathbf{G}_q(\mathbf{q})$  represents physically the weight or number of phonons of slowness vector  $\mathbf{q}$  that the solid can provide. If, instead of the imaginary part on the RHS of eq (11), we take the real part, use the delta function of  $t-\mathbf{q}\cdot\mathbf{x}$  instead of its derivative, and remove  $H(t)$ , then eq (10) is the Radon transform of the elastodynamic Green's function. The Radon representation of the Green's function has been developed by Wang and Achenbach [7] and applied to several interesting cases.

Using the representation of the delta function, we obtain from eq (11)

$$\mathbf{G}_q(\mathbf{q}) = -\pi \sum_s \mathbf{e}_{si}(\mathbf{q}) \mathbf{e}_{sj}(\mathbf{q}) \delta [E_s^2(\mathbf{q}) - 1], \quad (13)$$

where  $\mathbf{e}_s(\mathbf{q})$  ( $s=1,2,3$ ) and  $E_s^2(\mathbf{q})$  are, respectively, the eigenvectors and eigenvalues of  $\Lambda(\mathbf{q})$ . The right hand side of eq (10) requires integration over three variables – the three components of  $\mathbf{q}$ . However, the integrand is a product of two delta functions. Hence the integration over any two of the three variables is done analytically simply by substituting for their values determined by the delta functions. Numerical integration is required over only one variable. Moreover, the integrand is not singular and does not contain oscillatory functions. Consequently the delta-function representation is computationally much more efficient relative to the Fourier representation in eq (7).

Finally, the solution of the homogeneous equation, which can be verified by direct substitution for arbitrary  $\mathbf{f}(\mathbf{q})$ , is given by

$$\mathbf{u}_H(\mathbf{x},t) = (4\pi^3)^{-1} \int \mathbf{G}_q(\mathbf{q}) \mathbf{f}(\mathbf{q}) \delta^{(1)}(t-\mathbf{q}\cdot\mathbf{x}) d\mathbf{q}. \quad (14)$$

The virtual force  $\mathbf{f}(\mathbf{q})$  has to be determined by imposing the boundary conditions. Application of eq (14) to calculate the elastic waveforms in anisotropic solids has been given in [3]. Equations (10) and (14) reduce to corresponding elastostatic Green's function in the limit  $t=+0$ .

## References

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