The Relationship Between the Kramers-Kronig Dispersion Relations and the Time-Causal Relations

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Outline

- Motivation
- History
- Applications
- Problems
- Relationship with the time-causal relations

Introduction

- The Kramers-Kronig relations are based upon the fundamental notions of *linearity* and *causality*.
- The relations connect the real and imaginary parts of a *susceptibility* function (or *causal transform*).
- The relations also provide a criterion for the causal consistency of a *measurement* or *model* of the propagation mechanisms of a medium.

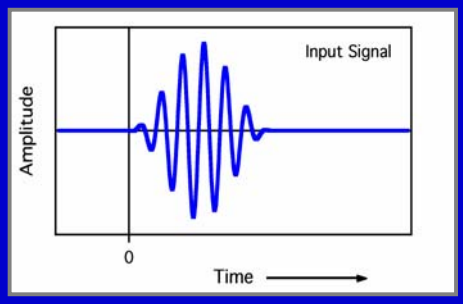
Motivation: Thought Experiment *

- Assume we have an 'ideal' material that is able to absorb the sound at only a single frequency without affecting the signal in any other manner.
- We will see that this suggests there is a relation between the absorption and change in phase as a signal propagates through a material.

*J. S. Toll, *Phys Rev*, 104, pp. 1760-1770, 1956.

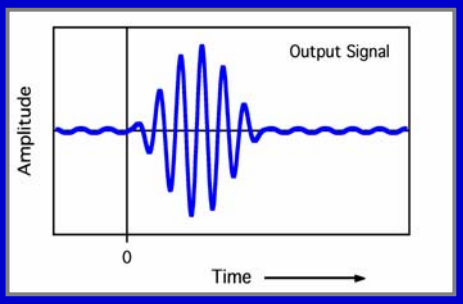
Motivation: 'Ideal' Material

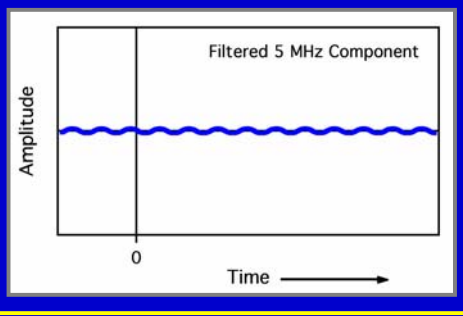
Incident Signal



Attenuate 5 MHz signal

Output Signal





Motivation: Observations

- The acausal aspect of this thought experiment is apparent from an output signal arriving prior to an input signal.
- To avoid the violation of causality, the attenuation at a particular frequency must be accompanied by a corresponding shift in the phases at all other frequencies, such that we observe destructive interference for $t < 0$.
- The relationship between the *attenuation* and *phase* can be determined from the appropriate dispersion relation.

History: Development

- Developed originally to relate the dispersion and absorption of x-rays from a linear and causal basis (1920s)
  - R. de Laer Kronig, *J. Opt. Soc. Am.* 12, 547-557 (1926).
  - H.A. Kramers, *Atti Congr. Intern. Fisici* 2, 545-557 (1927)
- Applied to a range of wave-based physics
  - Magnetic Resonance
  - Optical Spectroscopy
  - (Quantum) Particle Scattering

History: Applications to Acoustics

- Theoretical Description
  - V.L. Ginzberg, *Sov. Phys. Acoust.* 1, 32-41 (1955).
- Geophysics
  - W.I. Futterman, *J. Geophys. Research* 67, 5279-5291 (1962).
  - G.L. Lamb, *J. Geophys. Research* 67, 5273-5277 (1962).
- Underwater Acoustics
  - C.W. Horton, *J. Acoust. Soc. Am.* 55, 547-549 (1974).
- Ultrasonics
  - M. O'Donnell et al., *J. Acoust. Soc. Am.* 63, 1935-1937 (1978).

Review: Necessary Conditions*

- We assume that the system of interest is *linear* and *passive*.
- Assume also that the inverse Fourier transform ***g*(*t*)** of $\tilde{G}(\omega)$ vanishes for ***t* < 0.**
  $\Rightarrow$ Real and imaginary parts of $\tilde{G}(\omega)$ are related by the Hilbert transform

*part of Titchmarsh's theorem (See, for example, H.M. Nussenzveig, *Causality and Dispersion Relations* (Academic Press, New York, 1972).)

Review: Hilbert Transform Pair

The integral relationships between the real and imaginary parts of a causal transform $\tilde{G}(\omega)$ can be written as

$$\mathrm{Re}\tilde{G}(\omega) = -\frac{1}{\pi}\left[\mathrm{Im}\tilde{G}(\omega) * \mathcal{P}\left(\frac{1}{\omega}\right)\right]$$

$$\mathrm{Im}\tilde{G}(\omega) = +\frac{1}{\pi}\left[\mathrm{Re}\tilde{G}(\omega) * \mathcal{P}\left(\frac{1}{\omega}\right)\right]$$

Review: Kramers-Kronig Relations

The Hilbert transform pair can be written as integrals over only positive frequencies

$$\mathrm{Re}G(\omega) = \frac{2}{\pi}\mathcal{P}\int_0^\infty \frac{\omega' \,\mathrm{Im}G(\omega')}{\omega'^2 - \omega^2} d\omega'$$

$$\mathrm{Im}G(\omega) = -\frac{2\omega}{\pi}\mathcal{P}\int_0^\infty \frac{\mathrm{Re}G(\omega')}{\omega'^2 - \omega^2} d\omega'$$

Review: Acoustic Propagation

The propagation of acoustic waves can be described by the propagation term

$$e^{i\left(\tilde{K}(\omega)z - \omega t\right)}$$

where $\tilde{K}(\omega)$ is the complex acoustic wave number.

Review: Causal Transform

- A causal transform commonly used in ultrasonic measurements is the transfer function of a medium

$$\tilde{H}(\omega,d) = \exp\left[i\tilde{K}(\omega)d\right]$$

- Mathematically, it is more convenient * to consider the logarithm of the transfer function

$$\frac{\ln \tilde{H}(\omega,d)}{d} = i\tilde{K}(\omega) = -\alpha(\omega) + i\frac{\omega}{c(\omega)}$$

*R.E. Burge et al., *Proc. Roy. Soc. Lond.* A 350, pp. 191-212 (1976).

Review: Dispersion Relations

If we consider the complex *propagation factor*

$$\tilde{G}(\omega) \equiv i\tilde{K}(\omega) = -\alpha(\omega) + i\frac{\omega}{c(\omega)}$$

we can write* the following relation between the phase velocity and attenuation

$$\frac{\omega}{c(\omega)} = -\frac{1}{\pi}\left[\alpha(\omega) * \mathcal{P}\left(\frac{1}{\omega}\right)\right]$$

*K.R. Waters et al., *J. Acoust. Soc. Am.* 108(5), Pt. 1, 2114-2119 (2000).

Review: Approaches to Application

- **Differential form: Nearly-Local**
  - M. O'Donnell et al., *J. Acoust. Soc. Am.* 63(6), 1935-1937 (1978).
  - K.R. Waters et al., *IEEE Trans. Ultrason. Ferroelec. Freq. Contr.* 50(1), 68-76 (2003).
- **Integral form: Extrapolation**
  - K.R. Waters et al., *J. Acoust. Soc. Am.* 108(2), 556-563 (2000).
- **Integral form: Truncation**
  - J. Mobley et al., *J. Acoust. Soc. Am.*, 108(5), 2091-2106 (2000).

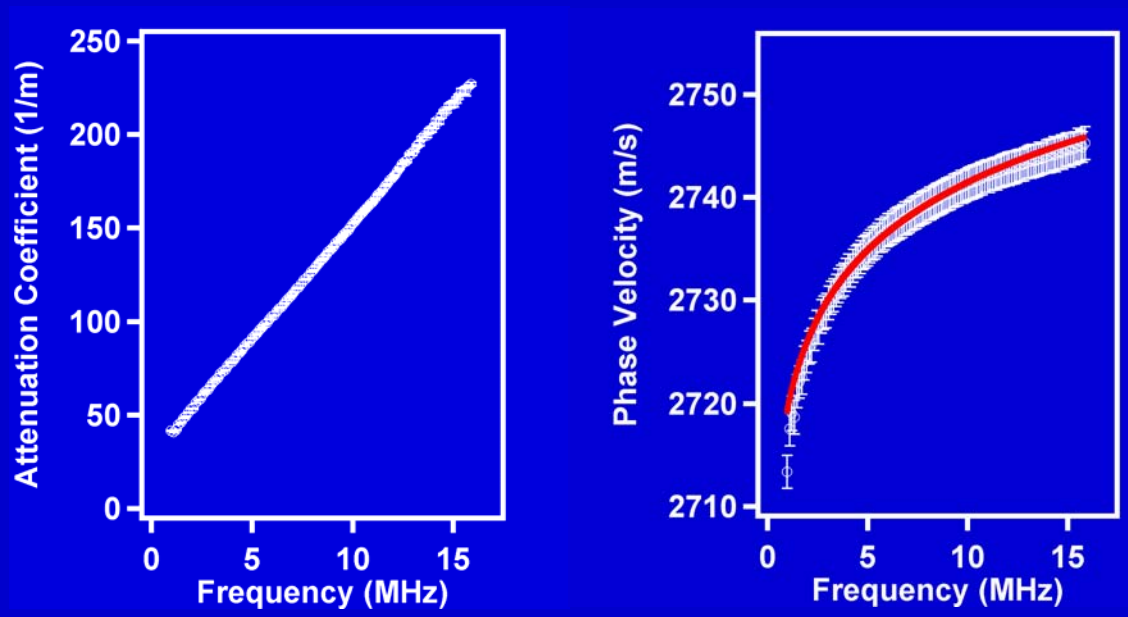
- **Time-Domain Convolution: Extrapolation**
  - T.L. Szabo, *J. Acoust. Soc. Am.*, 96(1), 491-500 (1994).

Linear-with-Frequency Attenuation

Assuming a linear-with-frequency attenuation model ($\alpha(\omega) = \alpha_0 \omega$), one finds for the K-K dispersion relation

$$\frac{1}{c(\omega)} - \frac{1}{c(\omega_0)} = -\frac{2}{\pi}\alpha_0 \ln\left|\frac{\omega}{\omega_0}\right|$$

PMMA: Propagation Properties



*K.R. Waters et al., *J. Acoust. Soc. Am.* 108(2), 556-563 (2000).

Nearly-Local Approximation

- The nearly-local approximation* provides a simple *differential relation* between the attenuation coefficient and phase velocity

$$\alpha(\omega) \approx -\frac{\pi}{2}\,\omega^2 \frac{d}{d\omega}\left(\frac{1}{c(\omega)}\right)$$

or

$$\frac{1}{c(\omega)} - \frac{1}{c(\omega_0)} \approx -\frac{2}{\pi}\int_{\omega_0}^{\omega} \frac{\alpha(\omega')}{\omega'^2}\,d\omega'$$

- However, it predicts *linear-with-frequency* dispersion for an ω^2 attenuation.

*M. O'Donnell, E.T. Jaynes, J.G. Miller, *J. Acoust. Soc. Am.*, 63(6), 1935-1937 (1978).

Dispersive Wave Equation

The wave equation proposed by Szabo* includes a convolution loss term $\mathcal{L}_\gamma(t)$ that accounts for both attenuation and dispersion directly in the time domain

$$\nabla^2 p(z,t) - \frac{1}{c_0}\frac{\partial}{\partial t}\left(\mathcal{L}_\gamma(t) * p(z,t)\right) - \frac{1}{c_0^2}\frac{\partial^2 p(z,t)}{\partial t^2} = s(z,t)$$

*T.L. Szabo, *J. Acoust. Soc. Am.*, 96(1), 491-500 (1994).

Time-causal approach

- The convolution loss term ($\mathcal{L}_\gamma$) is composed of an attenuation term ($\mathcal{L}_\alpha$) and a dispersive term ($\mathcal{L}_k$) that are related by the *time-causal relations* to insure causality.

$$\mathcal{L}_k = -i\,\mathrm{sgn}(t)\,\mathcal{L}_\alpha$$

$$\mathcal{L}_\alpha = +i\,\mathrm{sgn}(t)\,\mathcal{L}_k$$

- The attenuation part of the convolution loss term is given by the inverse Fourier transform of the attenuation coefficient.

Convolution Loss Term

For the case of power law attenuation $(1 < y < 2)$, the convolution loss term can be calculated as

$$\mathcal{L}_\gamma(t) = -2\mathbf{1}_+(\mathrm{t})\mathcal{F}^{-1}\{\alpha(\omega)\}$$

$$= -4\,y!\cos\left(\frac{\pi}{2}(y+1)\right)\frac{\mathbf{1}_+(\mathrm{t})}{|t|^{y+1}}$$

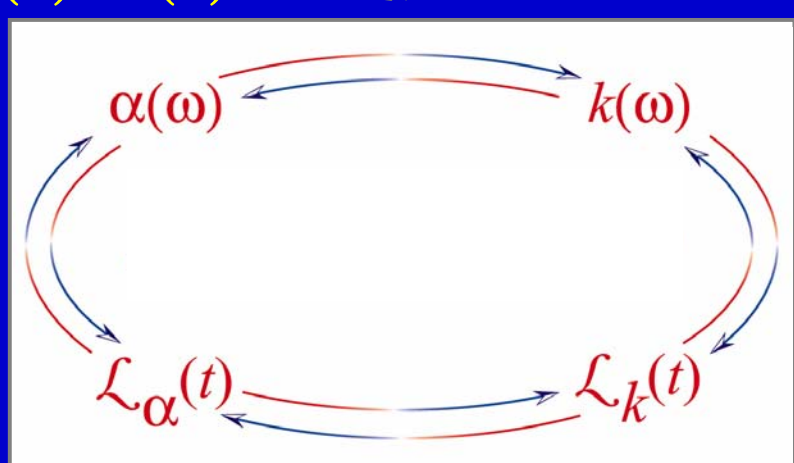
Recall

$$\nabla^2 p(z,t) - \frac{1}{c_0}\frac{\partial}{\partial t}\left(\mathcal{L}_\gamma(t) * p(z,t)\right) - \frac{1}{c_0^2}\frac{\partial^2 p(z,t)}{\partial t^2} = s(z,t)$$

Kramers-Kronig and Time-Causal

$\gamma(\omega) = -\alpha(\omega) + ik(\omega)$ $\quad\mathcal{H}\quad$ Kramers-Kronig

$\mathcal{F}$



$\mathcal{F}^{-1}$

Time Causal $\quad +i\,\mathrm{sgn}(t) \quad \mathcal{L}_\gamma(t) = \mathcal{L}_\alpha(t) + i\mathcal{L}_k(t)$

*K.R. Waters et al., *J. Acoust. Soc. Am.* 108(5), Pt. 1, 2114-2119 (2000).

Observations

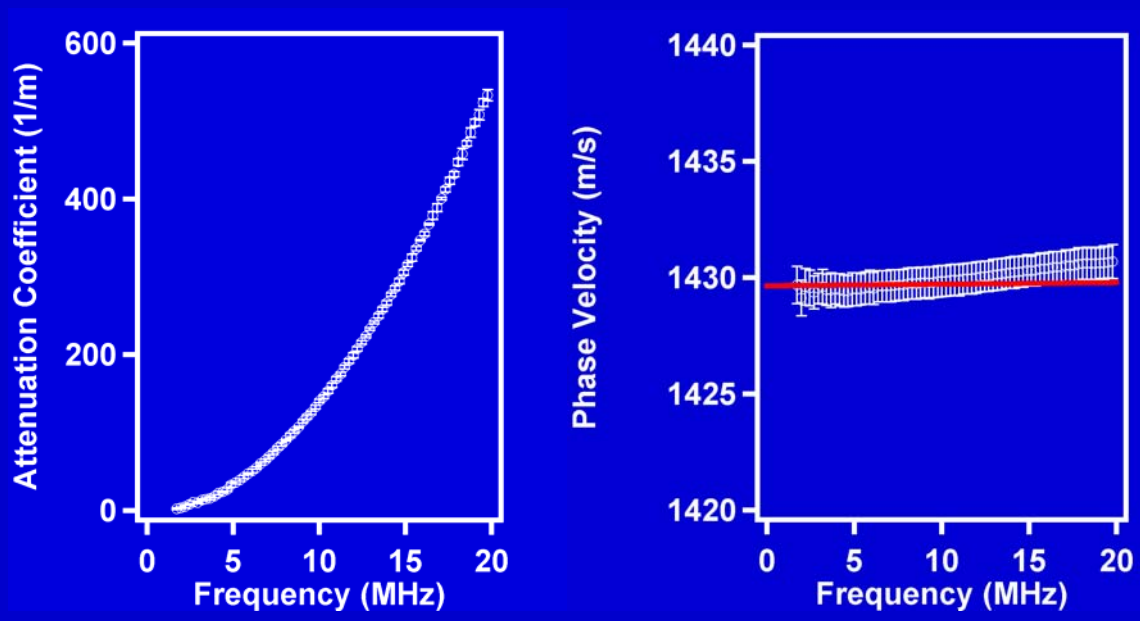
- The time-causal and Kramers-Kronig approaches to the study of dispersion are complementary methods.
- The time-domain convolution operator can be extended to other attenuation forms, namely those that have an inverse Fourier transform.

Media with Power Law Attenuation

Assuming a power law attenuation model ($\alpha(\omega) = \alpha_0 \omega^y$), one finds for the generalized Kramers-Krönig dispersion relation $(1 < y \leq 2)$

$$\frac{1}{c(\omega)} - \frac{1}{c(\omega_0)} = \alpha_0 \tan\left(\frac{\pi}{2} y\right)\left(\omega^{y-1} - \omega_0^{y-1}\right)$$

Silicone fluid: Attenuation and Dispersion

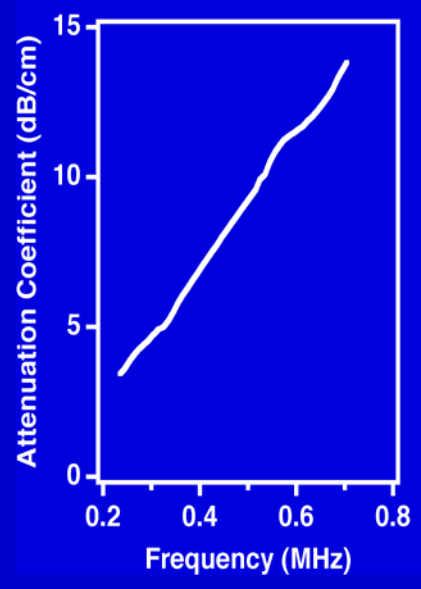


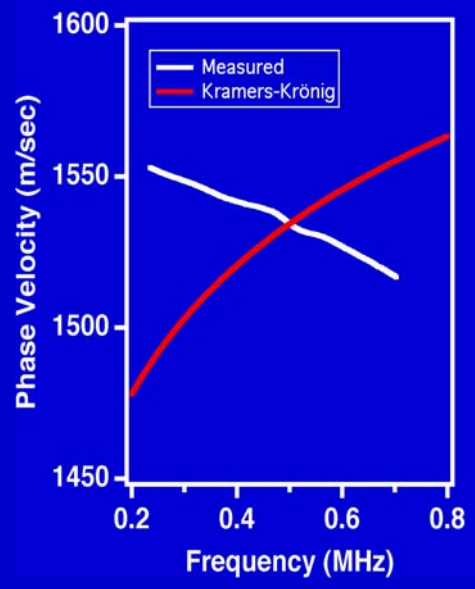
*K.R. Waters et al., J. Acoust. Soc. Am. 108(5), Pt. 1, 2114-2119 (2000).

Cancellous Bone: Inconsistency ?

- The Kramers-Kronig relations suggest that whenever the attenuation coefficient increases approximately linearly with frequency, then the phase velocity *increases* approximately logarithmically with frequency.
- Measurements from several laboratories on cancellous bone appear to be in conflict with this theoretical prediction.

Cancellous Bone: Measurements*





*P. Droin *et al.*, *IEEE UFFC*, 45(3), pp. 581-592 (1998).

Cancellous Bone: Consistency

- Agreement between experimental measurement and the K-K relations indicates a consistency with
  - the *linear* and *causal* basis of the K-K relations
  - the appropriateness of the chosen model
- A disagreement between experimental measurement and the K-K relations would suggest an *inconsistency* with the above condition(s).

Summary

- K-K relations are based on the notions of linearity and causality.
- K-K relations are useful for checking consistency of a model or measurement.
- Time-causal relations and K-K relations can be thought of as a Fourier transform pair.
- Time-causal relations and K-K relations are complementary approaches to the study of dispersion.