Uncertainty of Timebase Corrections

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***Abstract*—We develop a covariance matrix describing the uncertainty of a new timebase for waveform measurements determined with the National Institute of Standards and Technology's timebase correction algorithm. This covariance matrix is used with covariance matrices associated with other random and systematic effects in the propagation of uncertainty for the measured waveform.**

***Index Terms*—Covariance analysis, oscilloscopes, pulse measurements, waveform, uncertainty**

I. Introduction

THE National Institute of Standards and Technology (NIST) has developed a method for calibrating a waveform at every time point inside the waveform epoch [1]. The calibration also includes a covariance matrix that provides the uncertainty of each time point and the correlation between different time points. This covariance matrix is constructed from estimates of the random and systematic uncertainties of the measurement process. A detailed description of how a covariance matrix of the waveform is obtained, including both type A and type B evaluations of uncertainty, is given in [1] and [2].

One of the uncertainties we account for is the uncertainty associated with the use of a new timebase in the waveform. This new timebase is estimated by the NIST timebase correction procedure described in [3]. The timebase correction procedure corrects for jitter and timebase distortion in the waveform by use of orthogonal distance regression [4] to fit measurements of two quadrature sinusoids acquired simultaneously with the waveform being measured. We use the new timebase to interpolate the waveform to equally spaced time points. Consequently, the uncertainty of the interpolated waveform depends on the uncertainty of the estimate of the new timebase. In this paper we derive a formula for computing the uncertainties of the new timebase. This uncertainty is in the form of a covariance matrix.

The rest of the paper is organized as follows. Section II briefly reviews the procedure for timebase correction. In Section III we develop the formula for calculating the uncertainty covariance matrix of the new timebase. Section IV describes how to use the new timebase to interpolate the waveform and calculate its uncertainty. We conclude with some summary remarks in Section V.

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II. Timebase Correction

A generic system of the signal generator and sampling system for measuring and correcting timebase errors is described in [3]. We will describe the measurement apparatus that generates example waveforms used later in the paper to illustrate the calculation of the uncertainty of the new timebase. The measurement system and procedures are designed to produce two quadrature sinusoids and a waveform of interest having nearly identical timebase errors. The idea is that if we can estimate the timebase errors from the known sinusoidal signals, we can then apply our estimate to the waveform and compensate for timebase errors in its measurement.

Let y_{ij} denote the ith sample of the jth quadrature sinusoid ($j = 1$ and 2) measured at time t_{ij}. We use the following model [5] to describe the measurements of the two quadrature sinusoids:

$$y_{ij} = \alpha_j + \sum_{k=1}^{n_h} \left[\beta_{jk}\cos(2\pi k f t_{ij}) + \gamma_{jk}\sin(2\pi k f t_{ij})\right] + \epsilon_{ij}, \tag{1}$$

where $i = 1, 2, \ldots, n$, n_h is the harmonic order, f is the fundamental frequency of the waveforms, ϵ_{ij} is random additive noise, and α_j, β_{jk}, γ_{jk} are the parameters of the model. We write

$$t_{ij} = T_i + h_i + \tau_{ij},$$

where $T_i = (i-1)T_s$ is the target time of each sample with T_s being the target time interval between samples, h_i is the systematic timebase distortion, and τ_{ij} is the random jitter in each sampling time. With the assumption that the random jitter is common to all the waveforms, in particular to both sinusoids, we have $\tau_{i1} = \tau_{i2} = \tau_i$, and hence $t_{ij} = T_i + h_i + \tau_{ij} = T_i + h_i + \tau_i = T_i + \delta_i$. That is, $\delta_i = h_i + \tau_i$ is the timebase error at time T_i. With this simplification, we rewrite y_{ij}, given in (1), as a function F of $T_i + \delta_i$ and model parameters $\boldsymbol{\theta}_j = (\alpha_j,\, \beta_{j1}, \ldots \beta_{jn_h},\, \gamma_{j1}, \ldots,\, \gamma_{jn_h})$ as

$$y_{ij} = F(T_i + \delta_i;\, \boldsymbol{\theta}_j) + \epsilon_{ij}. \tag{2}$$

Estimates of timebase errors δ_i are readily available from the orthogonal distance regression fit of the model. In this approach, the model in (2) is fit to the data with the assumption that both the dependent (y_{ij}) and the independent (T_i) variables are subject to errors. (The errors in y_{ij} and T_i are ϵ_{ij} and δ_i, respectively.) The least-squares estimators of $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$, denoted by $\hat{\boldsymbol{\theta}}_1$, $\hat{\boldsymbol{\theta}}_2$, and $\hat{\boldsymbol{\delta}}$, are the solution of the minimization problem (with respect to $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, and $\boldsymbol{\delta}$) of the following function:

$$\begin{aligned} R(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\delta}) \quad = \quad & \sum_{i=1}^{n} \Big\{ w\left[F(T_i + \delta_i;\, \boldsymbol{\theta}_1) - y_{i1}\right]^2 + \\ & w\left[F(T_i + \delta_i;\, \boldsymbol{\theta}_2) - y_{i2}\right]^2 + \delta_i^2 \Big\}, \end{aligned} \tag{3}$$

where w is the weight used in the least-squares procedure. A detailed discussion on the use of the weight is given in [3]. An implementation of the above procedure using a freely available software ODRPACK [6] is discussed in [3] and is available at [7].

Once the $\hat{\delta}_i$ are obtained, we use

$$\hat{T}_i = T_i + \hat{\delta}_i \tag{4}$$

as the new timebase for the waveform that is measured simultaneously with the two quadrature sinusoids.

III. Uncertainty of the New Timebase Estimate

In this section we derive the formula for calculating the covariance matrix of the new timebase $\hat{T}_i$, which is equal to the covariance matrix of $\hat{\boldsymbol{\delta}}$. We first state a result from nonlinear regression analysis (e.g., see [8], p. 24) that we will use to derive the covariance matrix of $\hat{\boldsymbol{\delta}}$. Suppose that we have n observations $(\boldsymbol{x}_i,\, y_i)$, $i = 1, \ldots, n$, that follow a nonlinear model

$$y_i = F(\boldsymbol{x}_i,\, \boldsymbol{\theta}) + \epsilon_i, \tag{5}$$

where $\boldsymbol{\theta}$ is the unknown parameter and ϵ_i are zero-mean random noise with constant variance. Let $\hat{\boldsymbol{\theta}}$ be the least-squares estimate of $\boldsymbol{\theta}$ that minimizes the residual sum of squares

$$\sum_{i=1}^{n} [y_i - F(\boldsymbol{x}_i,\, \hat{\boldsymbol{\theta}})]^2 = \sum_{i=1}^{n} [g(y_i,\, \boldsymbol{x}_i,\, \hat{\boldsymbol{\theta}})]^2. \tag{6}$$

Then, under appropriate regularity conditions and for large n, the distribution of $\hat{\boldsymbol{\theta}}$ is approximately Gaussian, with mean $\boldsymbol{\theta}$ and covariance matrix

$$\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}} = \left(\boldsymbol{J}^T \boldsymbol{J}\right)^{-1} \sigma^2, \tag{7}$$

where σ^2 is the variance of ϵ_i, $\boldsymbol{J}$ is the Jacobian matrix whose $(i,\, j)$th element is given by

$$J_{ij} = \frac{\partial g(y_i,\, \boldsymbol{x}_i,\, \boldsymbol{\theta})}{\partial \theta_j}, \tag{8}$$

and $\boldsymbol{J}^T$ is its transpose. The derivatives are evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ and θ_j is the jth element of $\boldsymbol{\theta}$.

To apply the above result to our problem, we follow an approach described in [9] and rewrite the residual sum of squares in (3) as

$$R(\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2, \hat{\boldsymbol{\delta}}) = \sum_{i=1}^{3n} \left[g_i(\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2, \hat{\boldsymbol{\delta}}) \right]^2, \tag{9}$$

with

$$\begin{aligned} g_i &= \sqrt{w} \left[F(T_i + \hat{\delta}_i;\ \hat{\boldsymbol{\theta}}_1) - y_{i1} \right], \quad i = 1, \ldots, n \\ g_{i+n} &= \sqrt{w} \left[F(T_i + \hat{\delta}_i;\ \hat{\boldsymbol{\theta}}_2) - y_{i2} \right], \quad i = 1, \ldots, n \\ g_{i+2n} &= \hat{\delta}_i, \quad i = 1, \ldots, n. \end{aligned}$$

Let

$$\boldsymbol{\phi} = \begin{pmatrix} \hat{\boldsymbol{\theta}}_1 \\ \hat{\boldsymbol{\theta}}_2 \\ \hat{\boldsymbol{\delta}} \end{pmatrix} \tag{10}$$

be the column vector of size $2(2n_h + 1) + n$ containing the least-squares estimates of the parameters in (2). The (i, j)th element of the Jacobian matrix $\boldsymbol{J}$ is then given by

$$J_{ij} = \frac{\partial g_i}{\partial \phi_j}, \tag{11}$$

where ϕ_j is the jth element of $\boldsymbol{\phi}$. Since $J_{ij} = 0$ for the following cases:

$$\begin{aligned} i &= 1, \ldots, n \text{ and } \phi_j \in \hat{\boldsymbol{\theta}}_2, \\ i &= n+1, \ldots, 2n \text{ and } \phi_j \in \hat{\boldsymbol{\theta}}_1, \\ i &= 2n+1, \ldots, 3n \text{ and } \phi_j \in \hat{\boldsymbol{\theta}}_1, \\ i &= 2n+1, \ldots, 3n \text{ and } \phi_j \in \hat{\boldsymbol{\theta}}_2, \end{aligned}$$

we have

$$\boldsymbol{J} = \begin{pmatrix} \sqrt{w}\boldsymbol{A}_1 & \boldsymbol{0} & \sqrt{w}\boldsymbol{H}_1 \\ \boldsymbol{0} & \sqrt{w}\boldsymbol{A}_2 & \sqrt{w}\boldsymbol{H}_2 \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_n \end{pmatrix},$$

where

- $\boldsymbol{A}_1,\ n \times (2n_h + 1),$ is the Jacobian of g_i with respect to $\hat{\boldsymbol{\theta}}_1,\ i = 1, \ldots, n,$
- $\boldsymbol{A}_2,\ n \times (2n_h + 1),$ is the Jacobian of g_i with respect to $\hat{\boldsymbol{\theta}}_2,\ i = n+1, \ldots, 2n,$
- $\boldsymbol{H}_1,\ n \times n,$ is the Jacobian of g_i with respect to $\hat{\boldsymbol{\delta}}$, $i = 1, \ldots, n,$ and is diagonal, since $\partial g_i / \partial \hat{\delta}_j = 0$ if $i \neq j,$
- $\boldsymbol{H}_2,\ n \times n,$ is the Jacobian of g_i with respect to $\hat{\boldsymbol{\delta}}$, $i = n+1, \ldots, 2n,$ and is diagonal,
- $\boldsymbol{I}_n,\ n \times n,$ is the identity matrix.

The approximate covariance matrix $\boldsymbol{S}$ of $\boldsymbol{\phi}$ is then given by

$$\boldsymbol{S} = \left(\boldsymbol{J}^T \boldsymbol{J}\right)^{-1} \hat{\sigma}^2, \tag{12}$$

where $\hat{\sigma}^2$ is the residual variance of the least-squares fit and is given by $R(\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2, \hat{\boldsymbol{\delta}})/[n - 2(2n_h + 1)]$. Since $\boldsymbol{S}$ is the covariance matrix for all the parameter estimates of the model and we desire only the covariance matrix of $\hat{\boldsymbol{\delta}}$, we partition $\boldsymbol{S}$ as

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} \end{pmatrix}, \tag{13}$$

where $\boldsymbol{S}_{11}$, $(2n_h + 1) \times (2n_h + 1)$, is the covariance matrix for $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\theta}}_1\ \hat{\boldsymbol{\theta}}_2)^T$; $\boldsymbol{S}_{12} = \boldsymbol{S}_{21}^T$, $(2n_h + 1) \times n$, is the covariance matrix between $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\delta}}$; and $\boldsymbol{S}_{22}$, $n \times n$, is the covariance matrix for $\hat{\boldsymbol{\delta}}$. Now

$$\frac{\boldsymbol{J}^T \boldsymbol{J}}{w} = \begin{pmatrix} \boldsymbol{A}_1^T \boldsymbol{A}_1 & \boldsymbol{0} & \boldsymbol{A}_1^T \boldsymbol{H}_1 \\ \boldsymbol{0} & \boldsymbol{A}_2^T \boldsymbol{A}_2 & \boldsymbol{A}_2^T \boldsymbol{H}_2 \\ \boldsymbol{H}_1^T \boldsymbol{A}_1 & \boldsymbol{H}_2^T \boldsymbol{A}_2 & \boldsymbol{H}_1^T \boldsymbol{H}_1 + \boldsymbol{H}_2^T \boldsymbol{H}_2 + \frac{1}{w} \boldsymbol{I}_n \end{pmatrix}$$

Let

$$\begin{aligned}
\boldsymbol{Q}_{11} &= \begin{pmatrix} \boldsymbol{A}_1^T\boldsymbol{A}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_2^T\boldsymbol{A}_2 \end{pmatrix} \\
\boldsymbol{Q}_{12} &= \begin{pmatrix} \boldsymbol{A}_1^T\boldsymbol{H}_1 \\ \boldsymbol{A}_2^T\boldsymbol{H}_2 \end{pmatrix} \\
\boldsymbol{Q}_{21} &= \boldsymbol{Q}_{12}^T \\
\boldsymbol{Q}_{22} &= \boldsymbol{H}_1^T\boldsymbol{H}_1 + \boldsymbol{H}_2^T\boldsymbol{H}_2 + \frac{1}{w}\boldsymbol{I}_n
\end{aligned}$$

be the submatrices of $\boldsymbol{J}^T\boldsymbol{J}$ partitioned according to $\boldsymbol{S}_{ij}$ in (13). Note that $\boldsymbol{Q}_{22}$ is diagonal since both $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are diagonal.

With this partitioning, (12) becomes

$$\begin{pmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} \end{pmatrix} = \frac{\hat{\sigma}^2}{w}\begin{pmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{pmatrix}^{-1}.$$

It can be shown that (e.g., [10], page 33)

$$\boldsymbol{S}_{22} = \frac{\hat{\sigma}^2}{w}\left(\boldsymbol{Q}_{22} - \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{21}^T\right)^{-1}. \tag{14}$$

The above evaluation of $\boldsymbol{S}_{22}$ involves the inversion of an $n\times n$ matrix. This may not be acceptable if n is large. However, since $\boldsymbol{Q}_{22}$ is diagonal, this special structure can be exploited to evaluate $\boldsymbol{S}_{22}$ more efficiently. Based on a result in [10] (page 33), we have

$$\left(\boldsymbol{Q}_{22} - \boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{21}^T\right)^{-1} = \boldsymbol{Q}_{22}^{-1} - \boldsymbol{Q}_{22}^{-1}\boldsymbol{Q}_{21}\boldsymbol{P}^{-1}\boldsymbol{Q}_{21}^T\boldsymbol{Q}_{22}^{-1},$$

where $\boldsymbol{P} = (\boldsymbol{Q}_{21}^T\boldsymbol{Q}_{22}^{-1}\boldsymbol{Q}_{21} - \boldsymbol{Q}_{11})$. The calculation of $\boldsymbol{S}_{22}$ now involves the inversion of an $n \times n$ diagonal matrix $\boldsymbol{Q}_{22}$, which can be easily obtained, and the inversion of a $2(2n_h+1)\times 2(2n_h+1)$ matrix $\boldsymbol{P}$. In most of our applications, $n_h = 3$, thus we are required only to invert a 14×14 symmetric matrix.

IV. Interpolated Waveform and Its Uncertainty

Given the waveform of interest $\{z_i;\, i = 1,\ldots,n\}$, and its corresponding new timebase $\hat{T}_i$, we obtain m equally spaced interpolated values of z_i and their uncertainties. We use $\boldsymbol{S}_t = \boldsymbol{S}_{22}$ to denote the covariance matrix of $\hat{T}_i$. We assume that $\hat{T}_1 < \hat{T}_2 < \cdots < \hat{T}_n$. (A programming note, $\hat{T}_i$ and corresponding z_i must be sorted before evaluating $\boldsymbol{S}_{22}$.) Let $\hat{z}_i$ be the ith *linearly* interpolated value on an equally spaced grid $\{x_i;\, i = 1,\ldots,m\}$, then

$$\begin{aligned}
\hat{z}_i &= \frac{z_2 - z_1}{\hat{T}_2 - \hat{T}_1}(x_i - \hat{T}_1) + z_1 \quad \text{if } x_i \le \hat{T}_1 \\
&= \frac{z_{j+1} - z_j}{\hat{T}_{j+1} - \hat{T}_j}(x_i - \hat{T}_j) + z_j \quad \text{if } \hat{T}_j \le x_i \le \hat{T}_{j+1} \\
&= \frac{z_n - z_{n-1}}{\hat{T}_n - \hat{T}_{n-1}}(x_i - \hat{T}_{n-1}) + z_{n-1} \quad \text{if } \hat{T}_n \le x_i.
\end{aligned}$$

Let $\boldsymbol{S}_z$ denote the covariance matrix of the original waveform z_i, which can be obtained from repeated measurements or other means (see an example below). The covariance matrix of the interpolated waveform $\hat{z}_i$ is then given by

$$\boldsymbol{S}_{\hat{z}} \approx \boldsymbol{J}_t\boldsymbol{S}_t\boldsymbol{J}_t^T + \boldsymbol{J}_z\boldsymbol{S}_z\boldsymbol{J}_z^T, \tag{15}$$

where the (i,j)th elements of the Jacobian matrices $\boldsymbol{J}_t$ and $\boldsymbol{J}_z$ are $\partial\hat{z}_i/\partial\hat{T}_j$ and $\partial\hat{z}_i/\partial z_j$, respectively. Since there are only two non-zero elements on each row of $\boldsymbol{J}_t$ and $\boldsymbol{J}_z$, this simplifies the calculation of $\boldsymbol{S}_{\hat{z}}$. Further simplification can be realized if $\boldsymbol{S}_z$ is diagonal.

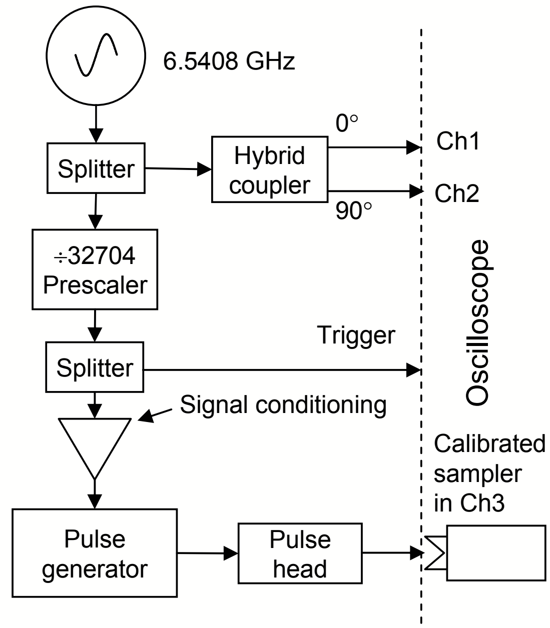

Fig. 1. Schematic diagram of the measurement apparatus.

We use the following experiment to illustrate the calculation of $\hat{z}_i$ and $\boldsymbol{S}_{\hat{z}}$. Figure 1 shows the measurement apparatus used to generate the waveforms. The synthesized signal generator produces 6.5408 GHz sine waves. The 90° hybrid coupler produces quadrature sinusoids that are measured on channels 1 and 2. The samplers in channels 1 and 2 have a nominal bandwidth of 12.4 GHz and a root mean square additive voltage noise (with no signal applied) of 0.2 mV. The prescaler produces a fast transition at nearly the maximum rate at which the pulse generator will trigger (200 kHz). This transition is used to trigger the oscilloscope, and a replica of the transition is delayed, passed through a limiting amplifier, and then used to trigger the pulse generator, which is measured simultaneously on channel 3. The generated pulse has a nominal 10 % to 90 % transition duration of 14.9 ps and the sampler of channel 3 has roughly a 70 GHz bandwidth with root mean square additive noise of 1.1 mV (with no signal applied). The internal architecture of the (commercially available) sampling oscilloscope [3] [11] is such that the trigger pulse starts the timebase of the oscilloscope. After a programmed delay, the timebase fires a single strobe pulse that is split to trigger all three of the samplers. Since the dominant timing errors in the measurement are due to the process of triggering the timebase and to the timebase itself, characterization of the timing errors in the sinusoids is a good estimate of the timing errors in the pulse generator. Additional sinusoids are measured at 6.6 GHz and 6.5 GHz for the purpose of estimating the timebase distortion [5] [12] and using it as an initial guess for the timebase error. Criteria for selecting the frequencies of the sinusoids are given in Section IV of [3].

In a typical experiment, waveforms are transferred to an external computer for post processing. We measure 100 quadrature sinusoids and waveforms of interest at desired frequency (6.5408 GHz in this example) on channels 1, 2, and 3. We also measure 10 additional quadrature sinusoids at each of the two different frequencies (6.6 GHz and 6.5 GHz in this example). We first estimate the timebase distortion based on all sinusoids and use it as an initial guess for the timebase error. Then, for each of the 100 sets of waveforms, we estimate the timebase error based on the sinusoids measured on channels 1 and 2 and use the new timebase to interpolate the waveform of interest measured on channel 3 to equally spaced time points. The covariance matrix of the interpolated waveform is also obtained using (15). Consequently, 100 interpolated waveforms and associated covariance matrices $\boldsymbol{S}_{\hat{z}}$ are calculated. The mean of the 100 waveforms is taken to be the measured waveform. We calculate two sources of uncertainty for the mean waveform. The first source of uncertainty is based on the "pooled" result of the 100 covariance matrices $\boldsymbol{S}_{\hat{z}}$. The second source of uncertainty is the variation among the 100 interpolated waveforms. We combine these two uncertainties to obtain the covariance matrix of the mean waveform.

The time-measurement window (waveform epoch) for the experiment is 5.1 ns with 4096 samples. The waveform is interpolated on 2048 evenly spaced time points from 0.02 to 5.02 ns. For illustration, we report only the results corresponding to the first set of measurements. In this set of measurements, the weight used in the timebase correction procedure is $w = 2.397$. With this value of w, the weighted residual sums of squares corresponding to ϵ (the first two terms on the right-hand side of (3)) and δ (the third term on the right-hand side of (3)) are 0.0029 and 0.0027, respectively. The residual standard error $\hat{\sigma}$ in (12) is estimated by $\sqrt{(0.0029 + 0.0027)/(4096 - 14)} = 0.0012$, where 14 is the number of parameters in $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ of (3).

Figure 2 displays the interpolated waveform (top panel) and its uncertainty (bottom panel). The uncertainties are the square root of the diagonal elements of the covariance matrix $\boldsymbol{S}_{\hat{z}}$, which is obtained by use of (15) with $\boldsymbol{S}_z = \hat{\sigma}_\epsilon^2 \boldsymbol{I}$, where $\hat{\sigma}_\epsilon$ is the estimate of the standard deviation of the distribution of the random additive noise ϵ_{ij} in (1) and is given by $\sqrt{0.0029/4082} = 0.00084$ in this example.

Once the covariance matrix of the waveform is obtained, it can be used, for example, to propagate uncertainties in the estimated pulse parameters of the waveform, such as the state levels, amplitude, and transition duration [13].

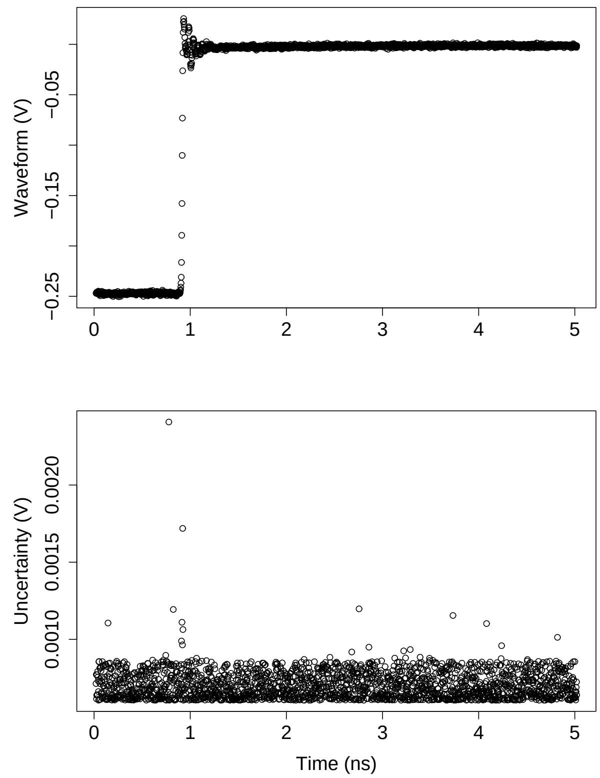

Fig. 2. An interpolated voltage step and its uncertainty.

V. Conclusion

In this paper we have presented an efficient method for calculating the covariance matrix of the new timebase of a waveform. We showed how to use this covariance matrix to propagate the uncertainty of an interpolated waveform with evenly spaced time points.

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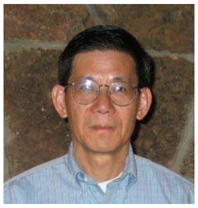

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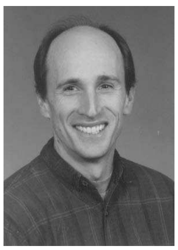

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