

# Adaptive Characterization of Jitter Noise in Sampled High-Speed Signals

Kevin J. Coakley, C.-M. Wang, Paul D. Hale, *Senior Member, IEEE*, and Tracy S. Clement, *Member, IEEE*

**Abstract**—We estimate the root-mean-square (RMS) value of timing jitter noise in simulated signals similar to measured high-speed sampled signals. The simulated signals are contaminated by additive noise, timing jitter noise, and time shift errors. Before estimating the RMS value of the jitter noise, we align the signals (unless there are no time shift errors) based on estimates of the relative shifts from cross-correlation analysis. We compute the mean and sample variance of the aligned signals based on repeated measurements at each time sample. We estimate the derivative of the noise-free signal based, in part, on a regression spline fit to the average of the aligned signals. Our initial estimate of the RMS value of the jitter noise depends on estimated derivatives and sample variances at time samples where the magnitude of the estimated derivative exceeds a selected threshold. This initial estimate is generally biased. Using a parametric bootstrap approach, we adaptively adjust this initial estimate of the RMS value of the jitter noise based on an estimate of this bias. We apply our method to real data collected at NIST. We study how results depend on the derivative threshold.

**Index Terms**—Adaptive, bias-correction, bootstrap, derivative estimation, high-speed, jitter, optoelectronics, regression spline.

## I. INTRODUCTION

**I**DEALLY, one would like to sample a signal at equally spaced time intervals. However, in high-speed measurement systems, the target time and actual sampling time may differ because of both systematic and random errors. The systematic component of this difference is time base distortion (TBD) [1]–[8]. We decompose the random timing error into a slowly varying component called drift, and a quickly varying component called jitter [9]–[12] (the terms “jitter” and “jitter noise” are synonymous and should not be confused with deterministic jitter). Within the time window during which a particular realization of the signal is sampled, the drift error manifests itself by shifting the waveform, while jitter manifests itself by perturbing each of the sampling times in the window by a random amount. Within the measured time window, each of the jitter noise realizations is independent of all the other jitter noise realizations. By definition, the expected value of a realization of the jitter noise at any time is 0. Thus, the root-mean-square (RMS) value of the jitter noise, that is the RMS jitter, equals the standard deviation of the jitter noise. Further, signal measurements are contaminated by additive noise. In this work, we estimate the RMS value of timing jitter

noise. Our work is motivated by efforts to characterize the impulse response functions of high-speed sampling oscilloscopes [13]–[16] that are contaminated by jitter noise (as well as by TBD errors, time shift errors, and additive noise).

For the case where the noise-free signal is a sinusoid plus its harmonics, there are methods to estimate the RMS value of the jitter noise [8], [17]. Here, we consider the more complex case where there is no analytical model for the signal. Hence, the methods developed in [8], [17] do not apply to the signals considered in this work. In principle, the RMS value of the jitter noise can be estimated from data collected over a time interval where the noise-free signal is a linear function of time [10]. If this linearity assumption is violated, estimates based on the premise of signal linearity are, in general, biased. Given the power spectrum of the jittered signal and a parametric model for the power spectrum of the jitter probability density function (pdf), it is possible to estimate the model parameters that characterize the power spectrum of the jitter pdf [9]. Since this approach is a nonlinear parameter estimation scheme, it is likely to be biased since estimates that are nonlinear functions of observed data are generally biased. For more discussion on the bias of nonlinear estimates, see page 40 of [18].

In this work, we estimate the RMS value of timing jitter noise based on a fully empirical estimate of the time-varying variance of the signal, and an estimate of the time-varying derivative of the unknown noise-free signal. We obtain the estimate of the signal derivative using a regression spline model [19] for the signal. In a regression spline model, one can model a signal without having a closed form analytical model for the signal. Our work is inspired by an earlier attempt to estimate the RMS value of jitter noise in high-speed sampled signals using a regression spline method [20]. For a clear discussion of regression spline modeling and related techniques, we direct readers to [21]. In [20], repeated measurements of jittered signals provided estimates of the sample variance of the signal for each time sample. Based on an estimate of the derivative of the noise-free signal provided by a regression spline model, and the sample variance of the signal at each time sample, they estimated the RMS value of the timing jitter noise. In their regression spline approach, the noise-free signal of interest was approximated as piecewise cubic polynomials. The regression spline parameters that define each cubic polynomial in each subinterval, the RMS value of the jitter noise, and the RMS value of the additive noise were jointly estimated using an iterative algorithm. However, their approach yielded biased estimates of the RMS value of the jitter noise.

Our implementation of the regression spline method differs from the implementation in [20]. First, we estimate the RMS

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K. J. Coakley and C.-M. Wang are with the Statistical Engineering Division, National Institute of Standards and Technology, Boulder, CO 80305 USA.

P. D. Hale and T. S. Clement are with the Optoelectronics Division, National Institute of Standards and Technology, Boulder, CO 80305 USA.

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jitter from a subset of the full signal, whereas in [20] they estimated RMS jitter from the full signal. To belong to this subset, the magnitude of the estimated derivative of the signal average must exceed a chosen threshold. We subsample because we expect that the least informative part of the signal corresponds to time samples where the magnitude of the derivative of the noise-free signal is least. Second and most significantly, we estimate the bias, that is the systematic error, of our estimate using a parametric bootstrap method [22]. Based on this bias estimate, we adjust our estimate accordingly. Because we correct our estimate for bias, our procedure is adaptive.

In a Monte Carlo simulation experiment, we consider two cases. In one case, the signals are contaminated by jitter noise and additive noise, but not by time shift errors. In the second case the signals are contaminated by jitter noise, additive noise, and time shift errors. For the second case, we estimate relative time shift errors based on cross-correlation analysis ([23] and Appendix). Given the relative time shift estimates, we align the signals using a Fourier method ([23] and Appendix). We estimate the RMS value of jitter noise from the average of the aligned signals and the sample variance of the aligned signals.

For the simulated and real signals considered in this work, we will show that RMS jitter can be estimated well by selecting a threshold value so that the RMS jitter value estimate is computed from the main feature in the signal which is similar to a damped sinusoid. During the time where the main feature rises from a local minimum to a local maximum, the signal is sampled approximately seven to eight times. For the cases studied, the RMS timing jitter value is no greater than  $2 dt$  where  $dt$  is the interval between samples. In general, the accuracy of our RMS jitter estimation method degrades as the true RMS jitter value increases.

In Section II, we present details for calculating our jitter estimate. In Section III, for simulated data, we study the performance of our bias-corrected (and uncorrected) jitter estimate based on the regression spline approach. We compare the performance of our bias-corrected estimate to the performance of two other estimates. One of the alternative estimates is based on the assumption that the noise-free signal is a linear function of time in the three-point time neighborhood about the time sample where the sample variance of the signal is largest. The other estimate is the regression spline approach presented in [20]. In Section IV, we estimate the RMS value of jitter noise in experimental signals measured at NIST.

## II. ESTIMATION PROCEDURE

### A. Preliminaries

Assuming that TBD errors are negligible, we can model the  $j$ th observed signal at the  $k$ th time sample as  $s_{jk}$  where

$$s_{jk} = f(t_k + \delta_j + \tau_{jk}) + \epsilon_{jk} \quad (1)$$

$\tau_{jk}$  is a realization of the jitter noise,  $\delta_j$  is a random time shift (drift) error,  $\epsilon_{jk}$  is a realization of additive noise, and  $f(\cdot)$  is an unknown function of time. For each signal, the realizations of the additive noise, jitter noise, and time shift noise processes are assumed to be independent. We assume that the expected value

of each is zero, that is,  $\langle \epsilon_{jk} \rangle = \langle \tau_{jk} \rangle = \langle \delta_j \rangle = 0$ , and that the realizations of the additive noise and jitter processes have finite variances  $\sigma_\epsilon^2$  and  $\sigma_\tau^2$ . We estimate the relative time shift errors by an all-pairs cross-correlation method ([23] and Appendix). Based on the estimated relative time shift errors, we translate each signal in time using a Fourier method ([23] and Appendix).

We denote the  $k$ th time sample of the  $j$ th aligned signal as  $s_{jk}^c$ . Assuming that we have accurately aligned the signals, a first order Taylor series argument yields an approximation for the variance of the sampled signal

$$\sigma_s^2(k) = \text{VAR}(s_{jk}^c) \approx |f'(t_k)|^2 \sigma_\tau^2 + \sigma_\epsilon^2 \quad (2)$$

(Here, we consider the general case where we do not have an analytic model for the derivative of the noise-free signal at time  $t_k$ ,  $f'(t_k)$ ). Hence, we must estimate this derivative from measurements). Using (2), we approximate the jitter variance,  $\sigma_\tau^2$  as

$$\sigma_\tau^2 \approx \frac{\sigma_s^2(k) - \sigma_\epsilon^2}{|f'(t_k)|^2}. \quad (3)$$

At any time sample of interest, we cannot determine the exact value of the variance of the signal from a finite number of signal measurements. Following the standard rule of statistical practice, we estimate the unknown theoretical variance by computing the empirical sample variance. As a first step in calculating the sample variance, we compute the average of the  $N$  aligned noisy signals as follows:

$$\bar{s}_k = \frac{1}{N} \sum_{j=1}^N s_{jk}^c. \quad (4)$$

The sample variance of an aligned signal at the  $k$ th time sample

$$\widehat{\sigma_s^2}(k) = \frac{1}{N-1} \sum_{j=1}^N (s_{jk}^c - \bar{s}_k)^2 \quad (5)$$

is an estimate of the theoretical variance  $\sigma_s^2(k)$  of the signal at the  $k$ th time sample. As the number of samples  $N$  approaches infinity, the sample variance converges to the theoretical variance. The square root of the sample variance,  $\widehat{\sigma_s}(k)$  is the estimated standard deviation of the signal at the  $k$ th time sample. Throughout this paper, we refer to the estimated standard deviation as the standard error.

### B. Naive Estimate

In order to exploit (3) to estimate the RMS value of the jitter noise, we need an estimate of the derivative of the noise-free signal and an estimate of the additive noise variance. In this work, we estimate the additive noise variance from sample variances computed near the time boundaries where the signal is flat. We define the time sample where the sample variance of the signal is largest to be the  $k^*$ th time sample. If we assume that the noise-free function is a linear function of time in the neighborhood of this time sample, the derivative of the noise-free signal is approximately equal to  $(\bar{s}_{k^*+1} - \bar{s}_{k^*-1})/(2 dt)$ . This derivative estimate is equal to the estimated slope determined by linear regression analysis. Based on this estimate of the derivative and

(3), a naive estimate of the standard deviation of the jitter noise is

$$\widehat{\sigma}_{\tau,n} = 2dt \sqrt{\frac{\widehat{\sigma}_s^2(k^*) - \widehat{\sigma}_\epsilon^2}{(\bar{s}_{k^*+1} - \bar{s}_{k^*-1})^2}} \quad (6)$$

where  $dt$  is the interval between time samples, and  $\widehat{\sigma}_\epsilon^2$  is our estimate of the additive noise variance.

Since the above naive estimate depends on data from just three time samples, we expect it to have a large variance compared to a judiciously constructed estimate that depends on data from more than three time samples. Moreover, we expect the naive estimate to be biased because 1) the estimated derivative may differ from the true derivative; 2) higher order terms in the Taylor series are neglected in (3); and 3) the (6) estimate is a nonlinear function of the observed data and nonlinear estimates are generally biased. For more discussion about the bias of non-trivial nonlinear estimates, see page 40 of [18]. The regression spline method estimate developed in this work is superior to the (6) estimate for two main reasons. First, we incorporate information from more than just three time samples. Second, we correct our estimate for bias, that is, systematic error. In the next section, we derive our first-pass (uncorrected) regression spline method estimate of RMS jitter. Afterward, we estimate the bias of this first pass estimate and then adjust the first pass estimate accordingly.

### C. Regression Spline Estimate: Uncorrected

In a regression spline approach, one selects a sequence of interior knots that partition the time interval during which the signal is measured into contiguous subintervals. Within any of these subintervals, the regression spline prediction of the signal of interest is a polynomial function of time. The coefficients of the polynomial vary from subinterval to subinterval. In [20], and in our work, we choose a cubic polynomial within each interval. For the cubic case, there are  $K$  interior knots and four exterior knots (two at each time boundary). Overall, the cubic regression spline model has  $K + 4$  independent basis functions. The  $K + 4$  regression spline model parameters are determined by the standard method of weighted least-squares where the weight at a particular time sample is inversely proportional to the sample variance at that time sample. In [20] as well as in our work, we use B-spline basis functions to represent the polynomials in each subinterval. For the cubic case, the first and second derivatives of the regression spline are continuous at the knots. Based on the regression spline model parameters, one can estimate the derivative of the noise-free signal as a function of time. This derivative information along with empirical estimates of the standard deviation of the jittered signal at each time sample, allow one to estimate the RMS timing jitter noise. The B-spline representation at time  $t$ , is denoted as  $s_b(t)$ , and the derivative of the B-spline representation at time  $t$  is denoted as  $s'_b(t)$ .

To compute our jitter estimate, it is convenient to define the following quantities at the  $k$ th time sample

$$a_k = \widehat{\sigma}_s^2(t_k) - \widehat{\sigma}_\epsilon^2 \quad (7)$$

and

$$b_k = |s'_b(t_k)|^2. \quad (8)$$

Given (3), we expect the ratio of  $a_k$  and  $b_k$  to be a rough estimate of the jitter variance at all  $k$ . Intuitively, we expect more information in  $(a_k, b_k)$  data at time samples where the magnitude of the derivative is relatively large. In our studies, the ratio  $r_k = a_k/b_k$  had a very large variance at time samples where the magnitude of the signal derivative was very small. Thus, the average of all the  $r_k$  values would be a poor estimate of the jitter variance. To reduce the influence of noisy  $(a_k, b_k)$  pairs on our estimate, we take two actions. First, we design our estimate so that it depends on  $(a_k, b_k)$  values at time samples where the magnitude of the estimated derivative is greater than a selected threshold. Second, we estimate the jitter noise variance as the ratio of the pooled  $a_k$  data and the pooled  $b_k$  data. Pooling is a natural way to reduce the influence of highly variable, i.e. non-informative,  $(a_k, b_k)$  values on the estimate. For a discussion of data pooling in other statistical estimation problems, see [24], [25]. Finally, we require that our variance estimate be nonnegative. Thus, our (nonnegative) estimate of the variance of the jitter noise is

$$\widehat{\sigma}_\tau = \sqrt{\max\left(0, \frac{\sum_k a_k H(s'_b(t_k), \alpha)}{\sum_k b_k H(s'_b(t_k), \alpha)}\right)} \quad (9)$$

where

$$H(s'_b(t), \alpha) = \begin{cases} 1 & \text{if } |s'_b(t)| > \alpha \max(|s'_b|) \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

and  $\alpha$  is an adjustable threshold. Our estimate of the RMS value of the jitter noise is  $\widehat{\sigma}_\tau$  (above, we denote the maximum of 0 and  $x$  as  $\max(0, x)$ ).

By lowering the threshold, we incorporate more of the measured data into our estimate. However, if the threshold is too low, prediction error may increase if we incorporate too much noisy data with little or no additional information content. Since the optimal choice of the threshold is not obvious, we study how the choice of threshold affects results in a Monte Carlo simulation experiment. In general, for any choice of threshold, we expect that the above estimate is biased since it is a nonlinear function of the observed data (nonlinear estimates are generally biased [18]). Next, we describe how to correct our estimate for this bias.

### D. Regression Spline Estimate: Corrected

We estimate the bias of our estimate using a parametric bootstrap procedure [22]. The parametric bootstrap procedure is a Monte Carlo resampling scheme for simulating synthetic data based on the observed data. Based on the distribution of the (9) jitter estimates computed from the synthetic data, one can estimate the standard deviation of the (9) jitter estimate computed from the observed data. Further, one can estimate the bias of the estimate and correct it accordingly. In the bootstrap simulation model, the noise-free signal is equated to the regression spline model estimate of the average of the aligned observed signals  $s_b(t)$ . Like the observed data, the synthetic signals are corrupted

TABLE I

STATISTICAL PROPERTIES OF ESTIMATES OF THE RMS VALUE OF THE JITTER NOISE. FOR EACH DERIVATIVE THRESHOLD VALUE ( $\alpha$ ) AND TRUE RMS JITTER NOISE VALUE ( $\sigma_\tau$ ), WE DISPLAY THE ESTIMATED BIAS (BIAS), STANDARD ERROR (se), AND RMS PREDICTION ERROR (rms) OF THE UNCORRECTED (9) AND BIAS-CORRECTED (14) ESTIMATE COMPUTED FROM 100 RUNS OF A SIMULATION EXPERIMENT. IN EACH RUN, RMS JITTER IS ESTIMATED FROM 100 NOISY SIGNALS. FOR EACH RUN, WE ESTIMATE THE STANDARD DEVIATION OF THE ESTIMATE BY A BOOTSTRAP RESAMPLING SCHEME. WE LIST THE MEAN (COMPUTED FROM ALL 100 RUNS) VALUE OF THE BOOTSTRAP ESTIMATE (15) OF THE STANDARD DEVIATION OF THE ESTIMATE AS  $\widehat{se}_{boot}$ . WE NORMALIZE ALL STATISTICAL QUANTITIES BY THE INTERVAL BETWEEN TIME SAMPLES  $dt$ . IN THIS STUDY, THERE ARE NO TIME SHIFT ERRORS. FOR EACH SIMULATION RUN, THERE ARE 100 SIGNALS. WE LIST THE NAIVE ESTIMATE (6) FOR COMPARISON

$\sigma_\tau/dt$	$\alpha$	uncorrected			corrected			
		BIAS/dt	se/dt	rms/dt	BIAS/dt	se/dt	rms/dt	$\widehat{se}_{boot}/dt$
0.10	0.00	-0.0090( 45)	0.0449	0.0456	-0.0084( 56)	0.0559	0.0562	0.0400
0.10	0.01	0.0003( 10)	0.0099	0.0099	0.0010( 10)	0.0097	0.0097	0.0095
0.10	0.10	0.0001( 3)	0.0034	0.0034	0.0001( 3)	0.0034	0.0034	0.0031
0.10	0.50	0.0000( 4)	0.0036	0.0036	-0.0001( 4)	0.0036	0.0036	0.0034
0.10	0.99	-0.0026( 9)	0.0085	0.0089	-0.0016( 9)	0.0088	0.0089	0.0080
0.10	naive	0.0104(8)	0.0084	0.0134				
0.50	0.00	0.0090( 14)	0.0135	0.0162	0.0025( 14)	0.0137	0.0138	0.0140
0.50	0.01	0.0087( 11)	0.0112	0.0141	0.0020( 11)	0.0112	0.0114	0.0114
0.50	0.10	0.0062( 12)	0.0125	0.0139	-0.0004( 12)	0.0121	0.0120	0.0117
0.50	0.50	0.0036( 16)	0.0155	0.0159	0.0002( 15)	0.0152	0.0151	0.0142
0.50	0.99	-0.0132( 37)	0.0374	0.0395	-0.0087( 40)	0.0396	0.0403	0.0339
0.50	naive	0.0472(37)	0.0370	0.0598				
1.00	0.00	0.0583( 29)	0.0285	0.0649	-0.0008( 26)	0.0256	0.0254	0.0277
1.00	0.01	0.0565( 26)	0.0256	0.0619	-0.0013( 23)	0.0229	0.0229	0.0273
1.00	0.10	0.0498( 28)	0.0275	0.0568	-0.0027( 24)	0.0243	0.0243	0.0283
1.00	0.50	0.0237( 37)	0.0374	0.0441	0.0011( 35)	0.0353	0.0352	0.0347
1.00	0.99	-0.0227( 77)	0.0769	0.0798	-0.0151( 79)	0.0789	0.0800	0.0714
1.00	naive	0.08760(98)	0.0984	0.1314				
2.00	0.00	0.4381( 87)	0.0867	0.4465	-0.0953( 51)	0.0509	0.1079	0.0978
2.00	0.01	0.4293( 79)	0.0794	0.4365	-0.0903( 51)	0.0513	0.1038	0.0992
2.00	0.10	0.3763( 92)	0.0924	0.3874	-0.0712( 54)	0.0536	0.0890	0.1056
2.00	0.50	0.1840(146)	0.1459	0.2344	0.0344( 128)	0.1281	0.1320	0.1517
2.00	0.99	-0.0350(223)	0.2226	0.2243	0.0002( 245)	0.2454	0.2442	0.1849
2.00	naive	0.3233(366)	0.3661	0.4870				

by time shift errors, additive noise and jitter noise. In the simulation, the time shift parameters are equated to the estimated relative time shift parameters computed from the observed data. In the bootstrap procedure, we assume that jitter and additive noise are Gaussian random variables with expected values equal to 0 and variances equal to those estimated from the primary “observed” data. The number of signals in each bootstrap set is the same as the number of observed signals. Throughout this work, we simulate  $J = 30$  bootstrap replications of the observed data. Since the observed data consist of 100 repeat measurements of the noisy signal, each bootstrap replication consists of 100 realizations of a noisy signal.

More formally, the  $j$ th bootstrap replication of the observed signal at the  $k$ th time sample is  $s_{jk}^*$ , where

$$s_{jk}^* = s_b(t_{jk}^*) + \epsilon_{jk}^* \quad (11)$$

$$t_{jk}^* = t_o + (k-1)dt + \hat{d}_{j1} + \tau_{jk}^* \quad (12)$$

and  $t_o$  is a constant. Above,  $s_b(t)$  is the regression spline model estimate of the average of the aligned signals,  $dt$  is the nominal spacing between time samples,  $\hat{d}_{j1}$  is our estimate of the relative time shift of the  $j$ th signal with respect to the first signal (Appendix and [23]),  $\tau_{jk}^*$  is a simulated jitter noise realization,

and  $\epsilon_{jk}^*$  is a simulated additive noise realization. In the (12) simulation model, we assume that time base distortion is 0 at all times [If time base distortion is not 0, we would add a term equal to the estimated TBD to the right hand side of (12). Given that TBD is nonzero, the regression spline model  $s_b(t)$  would be fit to the unequally-spaced time series  $(t_k, \bar{s}_k)$  where  $t_k = t_o + (k-1)dt + \widehat{TBD}(k)$  where  $\widehat{TBD}(k)$  is the estimate of the TBD. For more details on this approach, see [15], [26]]. The realizations of the jitter noise and additive noise are mutually independent realizations of Gaussian random variables with standard deviations equal to the corresponding values computed from the observed signals ( $\widehat{\sigma}_\tau$  and  $\widehat{\sigma}_\epsilon$ ). The expected values of the simulated jitter noise and additive noise realizations are 0.

For each bootstrap replication of the observed data, we estimate relative time shift errors and align the signals using the same algorithms used for the observed data. We estimate a new set of regression spline model parameters, a new RMS additive noise value, and a new RMS jitter noise value  $\widehat{\sigma}_\tau^*$ . The bootstrap estimate of the bias of our jitter estimate is

$$\widehat{BIAS}_{boot} = \frac{1}{J} \sum_{j=1}^J \widehat{\sigma}_{\tau_j}^* - \widehat{\sigma}_\tau \quad (13)$$

TABLE II

STATISTICAL PROPERTIES OF ESTIMATES OF THE RMS VALUE OF THE JITTER NOISE. FOR EACH VALUE OF THE DERIVATIVE THRESHOLD ( $\alpha$ ) AND TRUE RMS JITTER NOISE VALUE ( $\sigma_\tau$ ), WE DISPLAY THE ESTIMATED BIAS (BIAS), STANDARD ERROR (se), AND RMS PREDICTION ERROR (rms) OF THE UNCORRECTED (9) AND BIAS-CORRECTED (14) ESTIMATE COMPUTED FROM 100 RUNS OF A SIMULATION EXPERIMENT. FOR EACH RUN, WE ESTIMATE THE STANDARD DEVIATION BY A BOOTSTRAP RESAMPLING SCHEME. WE LIST THE MEAN VALUE OF THE BOOTSTRAP ESTIMATE (15) OF THE STANDARD DEVIATION OF THE ESTIMATE AS  $se_{boot}$ . WE NORMALIZE ALL STATISTICAL QUANTITIES BY THE INTERVAL BETWEEN TIME SAMPLES  $dt$ . FOR EACH SIMULATION RUN, THERE ARE 100 SIGNALS. WE LIST THE NAIVE ESTIMATE (6) FOR COMPARISON. UNLIKE FOR THE TABLE I CASE, THE SIGNALS ARE MISALIGNED DUE TO TIME SHIFT ERRORS. WE ESTIMATE RELATIVE TIME SHIFT ERRORS BY A CROSS-CORRELATION METHOD, AND ALIGN THE SIGNALS BY A FOURIER METHOD

$\sigma_\tau/dt$	$\alpha$	uncorrected			corrected			
		$\overline{BIAS}/dt$	$se/dt$	$rms/dt$	$\overline{BIAS}/dt$	$se/dt$	$rms/dt$	$\overline{se}_{boot}/dt$
0.10	0.00	-0.0155( 47)	0.0470	0.0493	-0.0112( 62)	0.0620	0.0627	0.0414
0.10	0.01	-0.0066( 10)	0.0103	0.0122	0.0005( 10)	0.0103	0.0102	0.0110
0.10	0.10	-0.0075( 3)	0.0035	0.0082	-0.0003( 4)	0.0035	0.0035	0.0031
0.10	0.50	-0.0121( 3)	0.0035	0.0126	-0.0013( 4)	0.0039	0.0041	0.0032
0.10	0.99	-0.0200( 6)	0.0062	0.0209	-0.0041( 7)	0.0069	0.0080	0.0064
0.10	naive	-0.0019(8)	0.0083	0.0085				
0.50	0.00	-0.0145( 14)	0.0141	0.0202	0.0035( 15)	0.0146	0.0149	0.0140
0.50	0.01	-0.0163( 11)	0.0112	0.0197	0.0023( 12)	0.0115	0.0117	0.0107
0.50	0.10	-0.0262( 13)	0.0129	0.0292	-0.0012( 13)	0.0130	0.0130	0.0110
0.50	0.50	-0.0527( 15)	0.0154	0.0549	-0.0061( 16)	0.0165	0.0175	0.0132
0.50	0.99	-0.1001( 31)	0.0312	0.1048	-0.0282( 35)	0.0346	0.0445	0.0243
0.50	naive	-0.0300(39)	0.0387	0.0488				
1.00	0.00	0.0044( 29)	0.0290	0.0291	0.0113( 27)	0.0272	0.0294	0.0276
1.00	0.01	-0.0005( 27)	0.0271	0.0269	0.0091( 25)	0.0253	0.0267	0.0270
1.00	0.10	-0.0301( 30)	0.0297	0.0422	0.0026( 28)	0.0284	0.0284	0.0289
1.00	0.50	-0.1087( 41)	0.0407	0.1160	-0.0099( 43)	0.0432	0.0441	0.0347
1.00	0.99	-0.2259( 76)	0.0758	0.2382	-0.0696( 85)	0.0854	0.1098	0.0511
1.00	naive	-0.0821(101)	0.1014	0.1301				
2.00	0.00	0.2456( 91)	0.0913	0.2619	0.0448( 65)	0.0654	0.0790	0.1099
2.00	0.01	0.2165( 90)	0.0902	0.2344	0.0376( 69)	0.0691	0.0784	0.1107
2.00	0.10	0.0570(129)	0.1291	0.1405	0.0368( 109)	0.1085	0.1141	0.1286
2.00	0.50	-0.2838(169)	0.1687	0.3298	0.0048( 200)	0.2001	0.1992	0.1452
2.00	0.99	-0.5652(183)	0.1830	0.5938	-0.1960( 227)	0.2269	0.2989	0.1232
2.00	naive	-0.1127(686)	0.6359	0.6917				

where  $\widehat{\sigma}_{\tau j}^*$  is the estimate of RMS jitter computed from the  $j$ th bootstrap replication. Our bias-corrected estimate of RMS jitter noise is

$$\widehat{\sigma}_\tau^* = \widehat{\sigma}_\tau - \widehat{BIAS}_{boot} = 2\widehat{\sigma}_\tau - \frac{1}{J} \sum_{j=1}^J \widehat{\sigma}_{\tau j}^*. \quad (14)$$

The bootstrap estimate of the standard deviation of the uncorrected jitter estimate is

$$se_{boot} = \sqrt{\frac{1}{J-1} \sum_{j=1}^J \left( \widehat{\sigma}_{\tau j}^* - \frac{1}{J} \sum_{k=1}^J \widehat{\sigma}_{\tau k}^* \right)^2}. \quad (15)$$

### III. SIMULATION STUDY

#### A. Results

We quantify the performance of our estimate in a Monte Carlo experiment. In the simulation study, we equate the noise-free signal to the regression spline estimate of the average of aligned experimental signals collected in an experiment at NIST. Each simulated signal is sampled at 2048 times. The regression spline

model has 1028 knots. Between knots, the interpolating polynomial is cubic. In each of many runs of the simulation, we generate 100 synthetic signals that are contaminated by jitter noise, additive noise, and possibly time shift errors. In the first case (Table I), there are no time shift errors. For this case, we assume the signals are aligned, that is, we do not estimate relative time shift errors. In the second case (Table II), the signals are misaligned due to time shift errors. For this case, we estimate the relative time shifts and align the signals (Table II, Appendix). In case 2, we model the time shift errors as realizations of a Gaussian AR(1) process [27] where the autocorrelation at lag 1 is 0.5 and the standard deviation is 0.5  $dt$ . In Figs. 1 and 2, we display the average of 100 aligned noisy signals, the magnitude of the estimated derivative of the signal average based on the regression spline model, and the standard error of the aligned signals as a function of time. In Tables I and II, we list the standard error of the bias estimate in parentheses. For instance, 0.0003(10) signifies that the bias estimate and associated standard error are 0.0003 and 0.0010.

In the simulation experiments, the true value of the RMS value of the additive noise,  $\sigma_\epsilon = 0.000603794$ . The RMS value of the additive noise is estimated from the first 50 samples after

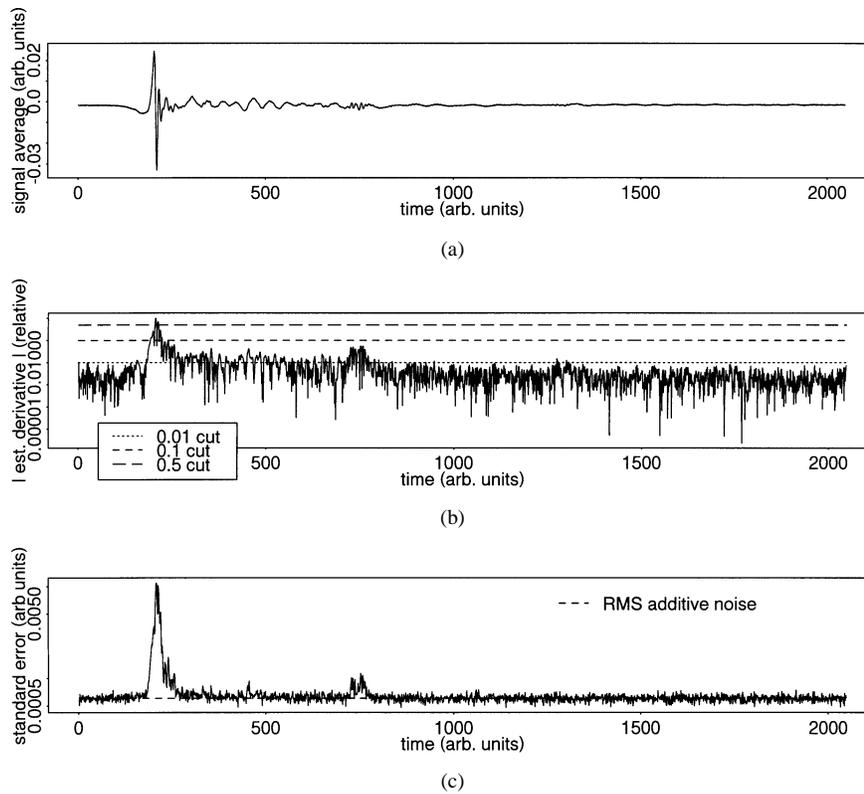


Fig. 1. (a) Signal average of 100 aligned simulated signals. (b) Magnitude of estimated derivative (scaled so that maximum derivative magnitude is 1). (c) Standard error of signal.

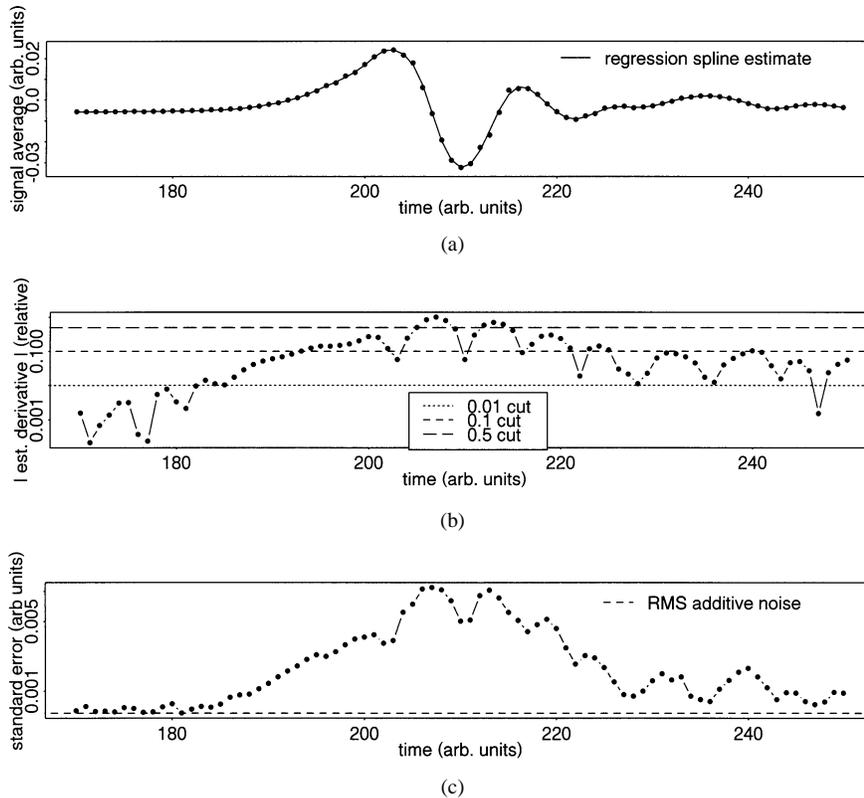


Fig. 2. (a) Signal average of 100 aligned simulated signals and regression spline estimate of signal average. (b) Magnitude of estimated derivative (scaled so that maximum derivative magnitude is 1). (c) Standard error of signal.

the 10th sample and the last 50 time samples before the 2039th time sample. We neglect the first 10 and last ten time samples

in order to suppress possible artifacts arising from boundary effects related to the Fourier algorithm.

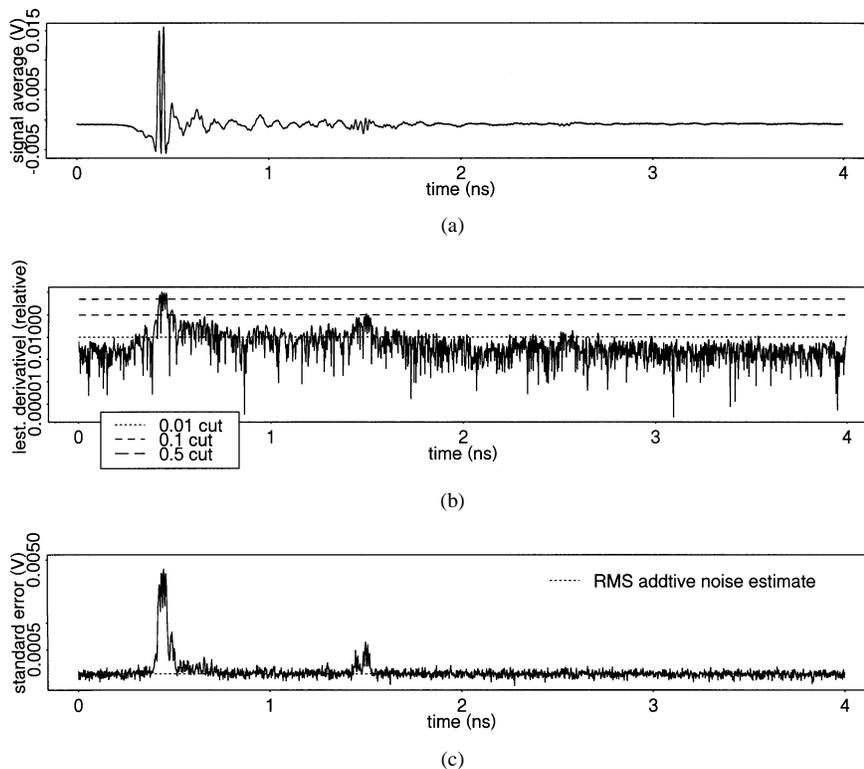


Fig. 3. (a) Signal average of 100 aligned observed signals. (b) Magnitude of estimated derivative (scaled so that maximum magnitude is 1). (c) Standard error of signal. The interval between samples is  $dt = 1.96$  ps. The RMS additive noise estimate is  $\sigma_\epsilon = 0.000268$  V.

### B. Comments

For the case of no time shift errors,  $\sigma_\tau = 0.5 dt$  and the same additive noise as before, we estimate the RMS value of the jitter noise by the computationally expensive method presented in [20]. In a study based on 500 Monte Carlo replications, the mean and standard deviation of the 500 estimates of the RMS value of the jitter noise are  $0.442 dt$  and  $0.012 dt$ , respectively. Thus, the bias of the method in [20] is larger than the corresponding bias of our estimate for this jitter noise level (see Table I).

In general, the RMS prediction error of the corrected estimate (14) is less than the RMS prediction error of the naive estimate (6) for all thresholds (Tables I, II). For most cases, the magnitude of the bias of the corrected estimate is closer to 0 than is the magnitude of the bias of the naive estimate. We conclude that the bias-corrected estimate (14) is superior to the naive estimate (6) of RMS timing jitter noise.

In general, the corrected estimate has lower RMS prediction error for the case where there are no time shift errors (Table I) compared to case where there are time shift errors (Table II). Since the relative time shift estimates are not perfect, we expect that jitter estimation is more difficult for the second case (Table II). For the four jitter noise levels considered in Table II, the RMS prediction errors [23] of the relative shift estimates are  $0.03$ ,  $0.16$ ,  $0.30$ , and  $0.68 dt$ . In general, as additive noise or jitter noise increases, the performance of our estimate should deteriorate.

The bias of the uncorrected estimate (9) depends strongly on threshold. However, in general, the bias of the corrected estimate

(14) is relatively small (relative to the standard error) and stable for thresholds equal to 0.01, 0.1, and 0.5. For both cases (drift and no drift), the bias of the corrected estimate is much lower than the bias of the uncorrected estimated.

The standard deviation of the estimate depends on the threshold (Tables I, II). Except for the lowest jitter case of  $\sigma_\tau = 0.1 dt$ , the highest threshold of  $\alpha = 0.99$  yields the estimate with the largest RMS prediction error and largest standard error. For  $\sigma_\tau = 0.1 dt$ ,  $\alpha = 0$  yields the estimate with highest RMS prediction error and largest standard error. In general, except for the lowest and highest thresholds, the mean bootstrap prediction of the standard deviation of the estimate is close to the actual standard error of the estimate. The bootstrap estimate of the standard deviation of the estimate is a promising diagnostic statistic for threshold selection. For instance, for any given data, we might select the threshold which minimizes  $\hat{s}_\epsilon$ .

In the parametric bootstrap procedure, we assume that the jitter pdf is Gaussian. For real applications, the actual jitter pdf may not be Gaussian. As the departure from normality becomes more extreme, the reliability of the bias estimate should deteriorate. To explore this issue, we simulated observed data contaminated by Gaussian timing jitter noise as before. However, when simulating bootstrap replications of the observed data, we sample timing jitter noise from a uniform distribution. The uniform distribution is centered on 0 so that the expected value of the jitter noise is 0 as before. Further, the upper and lower endpoints of the uniform distribution are selected so that the variance of the simulated jitter noise equals the estimated variance of the timing jitter noise. Except for the highest jitter case, the statistical properties of the corrected jitter estimate were almost

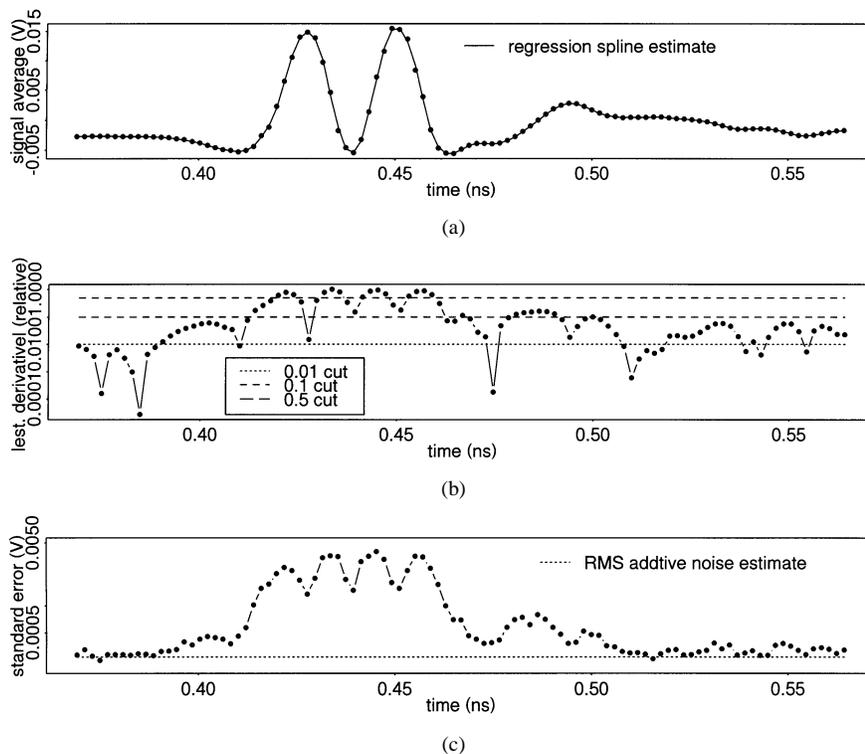


Fig. 4. (a) Signal average of 100 aligned observed signals and regression spline estimate. (b) Magnitude of estimated derivative (scaled so that maximum magnitude is 1). (c) Standard error of signal. The nominal interval between samples is  $dt = 1.96$  ps. The RMS additive noise estimate is  $\sigma_e = 0.000268$  V.

the same as before. For the highest jitter case, this modeling error inflated the RMS prediction error of the estimate by about 50%.

#### IV. REAL DATA

We estimate the RMS jitter from 100 misaligned measured high-speed sampled signals. We estimate RMS jitter as described before. Due to TBD errors, the nominal sampling times are not equally spaced. We fit the regression spline model to the signal average of the aligned signals. Based on the B-spline model parameters, we estimate the first derivative of the noise-free signal at each of 2048 unequally spaced time samples. The derivative estimates are computed in a way which accounts for TBD errors that are estimated in a separate experiment. (We fit the regression spline model to data of the form  $(t_k, \bar{s}_k)$ ,  $k = 1, 2, \dots$  where  $t_k$  is the sum of the  $k$ th nominal sampling time plus the value of the TBD at the  $k$  time sample. Given the estimated regression spline parameters, we can easily estimate the derivative at any  $t_k$  value.) In Figs. 3 and 4, we display the signal average of the 100 aligned signals, an estimate of the derivative of the noise-free signal, and the sample variance of the observed signals.

In Table III, we list estimates of the RMS value of jitter computed from observed data as a function of the derivative threshold  $\alpha$  and as a function of the number of knots. The uncorrected estimate of the RMS jitter noise depends strongly on threshold. However, the corrected estimate does not depend strongly on threshold. The highest values of the bootstrap estimates of the standard deviation of the corrected estimate were for the lowest and highest thresholds of 0 and 0.99. The variation of the corrected estimate with knot number, and with

threshold, is not large compared to the estimated random error ( $se_{boot}$ ). The stability of the jitter estimate as a function of threshold and number of knots supports the claim that our regression spline model is sufficiently complex for our purpose (of estimating RMS jitter noise).

In the bias-correction step, we assume that the actual jitter pdf is Gaussian. When the jitter is assumed to be uniform, the bias-corrected estimates are lower than the ones computed based on the Gaussian assumption by approximately 1%. In general, if the timing jitter noise pdf is not Gaussian, this sort of robustness analysis may help quantify systematic error not removed by our bias-correction procedure.

Between 0.42 ns and 0.47 ns, the signal has a quasisinusoidal form. Over this interval, the signal rises from a local minimum to a local maximum twice. During this rise, the signal is sampled about seven to eight times. Recall, that the simulated signals were sampled about the same number of times during their rise time. This gives us confidence that the sampling rate is high enough to accurately characterize the jitter noise.

All thresholds with the exception of  $\alpha = 0.99$  yield similar results. The largest threshold yields a result that appears to be inconsistent with the lower threshold results. This result is consistent with the simulation results. Based on this observation, the result for the highest threshold is not as trustworthy as the results for the lower thresholds.

#### V. CONCLUSION

We estimated the RMS value of timing jitter noise in simulated signals that were similar in complexity to high-speed sampled signals collected at NIST. We modeled the noise-free signal as a piece-wise polynomial using a regression spline approach.

TABLE III

WE ESTIMATE THE RMS VALUE OF TIMING JITTER NOISE BASED ON 100 ALIGNED EXPERIMENTAL SIGNALS. THE NOMINAL INTERVAL BETWEEN SAMPLES IS 1.96 ps. THE NUMBER OF SAMPLES PER SIGNAL IS 2048. BEFORE JITTER ESTIMATION, SIGNALS ARE ALIGNED. DERIVATIVE ESTIMATES ACCOUNT FOR TIME BASE DISTORTION ERRORS THAT ARE ESTIMATED IN A SEPARATE CALIBRATION EXPERIMENT. BOTH THE UNCORRECTED (9) AND BIAS-CORRECTED (12) ESTIMATE OF THE RMS VALUE OF THE TIMING JITTER NOISE ARE LISTED. WE ESTIMATE THE STANDARD DEVIATION OF THE ESTIMATE BY A PARAMETRIC BOOTSTRAP RESAMPLING SCHEME. THE BOOTSTRAP ESTIMATE OF THIS STANDARD DEVIATION IS LISTED AS  $se_{boot}$ . THE BOOTSTRAP ESTIMATE OF THE STANDARD DEVIATION IS AN APPROXIMATION TO THE STANDARD ERROR THAT WOULD HAVE BEEN COMPUTED FROM MULTIPLE REALIZATIONS OF THE OBSERVED DATA

no. knots	$\alpha$	samples above threshold	uncorrected est. (ps)	corrected est. (ps)	$se_{boot}$ (ps)
800	0.00	all	1.405	1.402	0.033
800	0.01	460	1.394	1.394	0.032
800	0.10	36	1.379	1.397	0.034
800	0.50	14	1.333	1.400	0.030
800	0.99	1	1.214	1.362	0.073
1028	0.00	all	1.394	1.393	0.032
1028	0.01	462	1.389	1.390	0.032
1028	0.10	34	1.367	1.386	0.033
1028	0.50	14	1.309	1.371	0.030
1028	0.99	1	1.140	1.260	0.073
1600	0.00	all	1.393	1.406	0.035
1600	0.01	574	1.389	1.405	0.035
1600	0.10	33	1.370	1.400	0.037
1600	0.50	14	1.314	1.401	0.040
1600	0.99	1	1.132	1.314	0.085

In one case, the signals were contaminated by additive noise and timing jitter noise. In the other case, the signals were contaminated by additive noise, jitter noise, and time shift errors. For the second case, we aligned the signals based on estimated values of the relative time shifts determined from cross-correlation analysis.

Based on repeated measurements of the noisy signals, we computed the sample variance of the (aligned) signals as a function of time. Based on a regression spline model, we estimated the derivative of the signal average at each time sample. Our RMS timing jitter estimate was computed from the estimated derivatives and sample variances at samples where the magnitude of the estimated derivative exceeded a selected threshold. Using a parametric bootstrap approach, we adjusted the estimate for bias. In general, the bias of the corrected estimate was much lower than the bias of the uncorrected estimate. For intermediate thresholds in range from 0.01 to 0.5, the bias of the corrected estimate was relatively small and stable for the simulated data. However, in general, our bias-correction scheme was not as effective when we selected the largest threshold of 0.99.

For real data, the uncorrected estimate of the RMS jitter noise depended strongly on threshold. However, the corrected estimate did not depend strongly on threshold. The bootstrap estimate of the standard deviation of the corrected estimate was lowest for thresholds between 0.01 and 0.5.

Provided that the signal is sampled at a sufficiently high rate, we expect our method to be valid for cases where the noise-free signal is well approximated as piece-wise cubic polynomials where the first and second derivatives are continuous. We recommend that users of our methods perform a stability study to verify that the sampling rate is sufficiently high for the purpose of estimating the RMS value of the jitter noise. We also recommend that users demonstrate that the regression spline has a sufficient number of knots in order to sufficiently model the complexity of the signal of interest. The user should verify that

the RMS jitter estimate stabilizes as the number of knots in the regression spline model increases.

## APPENDIX

### ALIGNMENT OF SIGNALS

Our approach for aligning signals generally follows the method presented in [23]. However, we implement a faster version of the approach described in [23]. In brief, we estimate the relative time shift of each distinct pair of signals by minimizing the mean square difference (MSD) between one signal and the shifted version of the other signal. We evaluate MSD at relative shifts which are integral multiples of the interval between samples  $dt$ . We then interpolate to estimate a relative shift which is, in general, a nonintegral multiple of  $dt$ . We combine relative shift estimates computed from all  $N(N-1)/2$  pairs of signals to estimate the  $N-1$  time shift parameters. We give the technical details of the approach below.

We assume that each noisy signal is shifted with respect to the others. That is, we model the expected value of the  $k$ th signal at time  $t$  as

$$\langle s^k(t) \rangle = \bar{s}(t + \delta_k) \quad (16)$$

where  $\delta_k$  is an unobserved time shift parameter and  $\bar{s}(t)$  is the unobserved reference signal. From a set of  $N$  signals, we cannot estimate the set of absolute time shifts  $\delta_1, \delta_2, \dots, \delta_N$ . Instead, we estimate relative time shift of the  $j$ th and first signal  $d_{j1} = \delta_j - \delta_1$ .

For the signal pair containing the  $i$ th signal and  $j$ th signal, we compute

$$\text{MSD}(\Delta) = \sum_k (s^i[t_k - \Delta] - s^j[t_k])^2. \quad (17)$$

as a function of  $\Delta$ . The translation  $\Delta$  is an integral multiple of the interval between samples  $dt$ . In the above expression,

we sum over all time samples except the first and last 20 time samples. In our approach, we assume that the signal is flat near the time boundaries. (The plateau levels at the beginning and end of the signal may not be exactly equal.) In our applications, the magnitude of the value of  $\Delta$  that minimizes MSD is less than  $20 dt$ .

We compute MSD in an neighborhood of a good initial guess for the optimal value of  $\Delta$ . Our initial guess is provided by a fast relative shift estimation method based on time centroids of the magnitude of the signal [23]. Our estimate of the time centroid of a signal  $s(t)$  is

$$\hat{C} = \frac{\sum_k |s(t_k)| H(t_k, \alpha) t_k}{\sum_k |s(t_k)| H(t_k, \alpha)} \quad (18)$$

where

$$H(t, \alpha) = \begin{cases} 1 & \text{if } |s(t)| > \alpha \\ 0 & \text{otherwise} \end{cases}. \quad (19)$$

We set  $\alpha = 0.5 \max(|s(t)|)$  for the signals studied in this work. For the signals studied here, this choice of  $\alpha$  is reasonable. In general, one might select the threshold  $\alpha$  using the adaptive technique in [23].

If the value of  $\Delta$  that minimized MSD is on a boundary of the search neighborhood, we repeat (recursively) the search for a neighborhood with twice as many points. For the simulated data (Section III), the initial search neighborhood had 17 points. For the real data (Section IV), the initial search neighborhood had 7 points. We define the value of  $\Delta$  on the lattice that minimizes MSD to be  $\Delta^*$ . We estimate the optimal value of  $\Delta$ ,  $\hat{\Delta}$ , by quadratic interpolation as follows:

$$\hat{\Delta} = \Delta^* - \frac{\text{MSD}(\Delta^* + dt) - \text{MSD}(\Delta^* - dt)}{2(\text{MSD}(\Delta^* + dt) + \text{MSD}(\Delta^* - dt) - 2\text{MSD}(\Delta^*))}. \quad (20)$$

We denote the (20) estimate of the relative shift of the  $i$ th and  $j$ th signals as  $\hat{\Delta}_{ij}$ . The complete cross-correlation method estimate of the relative time shift  $d_{k1}$  is

$$\hat{d}_{k1} = \frac{1}{N} \left( 2\hat{\Delta}_{k1} + \sum_{m \neq k, j} (\hat{\Delta}_{m1} - \hat{\Delta}_{mk}) \right). \quad (21)$$

We align the  $k$ th signal with respect to the first signal by translating it by the amount  $\hat{d}_{k1}$ . Since the translation is not an integral multiple of  $dt$ , we align each signal using a Fourier method. We compute the complex Fourier transform of the signal to be translated. We choose a frequency representation which ranges from 0 to twice the Nyquist frequency. At frequencies less than or equal to the Nyquist frequency, we multiply the complex Fourier transform by  $\exp(-2\pi f \hat{d}_{k1})$  where  $f$  is frequency. The Fourier transforms at the other frequencies above the Nyquist frequency are adjusted so that they satisfy a complex conjugate symmetry with the (adjusted) Fourier transform values at the corresponding frequencies below the Nyquist frequency. The translated version of the  $k$ th signal is equated to the real part of the inverse Fourier transform of the adjusted Fourier transform.

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**Kevin J. Coakley** received the Ph.D. degree in statistics from Stanford University, Stanford, CA, in 1989.

He is a Mathematical Statistician with the Statistical Engineering Division of the National Institute of Standards and Technology (NIST), Boulder, CO. His research interests include statistical signal processing, computer intensive statistical methods, and stochastic modeling, planning, and analysis of experiments in physical science and engineering.

**C.-M. Wang** received the Ph.D. degree in statistics from Colorado State University, Fort Collins, in 1978.

He is a Mathematical Statistician with the Statistical Engineering Division of the National Institute of Standards and Technology (NIST), Boulder, CO. His research interests include interval estimation on variance components, statistical graphics and computing, and the application of statistical methods to physical sciences.

Dr. Wang is a Fellow of the American Statistical Association.

**Paul D. Hale** (M'01–SM'01) received the Ph.D. degree in applied physics from the Colorado School of Mines, Golden, CO, in 1989.

He has been a Staff Member of the National Institute of Standards and Technology (NIST), Boulder, CO, since 1989 and has conducted research in birefringent devices, mode-locked fiber lasers, fiber chromatic dispersion, broadband lasers, interferometry, polarization standards, and high-speed optoelectronic measurements. He is presently Leader of the High-Speed Measurements Project in the Sources and Detectors Group. His research interests include high-speed optoelectronic and microwave measurements and their calibration.

Dr. Hale is currently an Associate Editor of the JOURNAL OF LIGHTWAVE TECHNOLOGY. Along with a team of four scientists, he received the Department of Commerce Gold Medal in 1994 for measuring fiber cladding diameter with an uncertainty of 30 nm. In 1998, he received a Department of Commerce Bronze Medal, along with four other scientists, for developing measurement techniques and standards to determine optical polarization parameters.

**Tracy S. Clement** (S'89–M'92) received the Ph.D. degree in electrical engineering from Rice University, Houston, TX, in 1993. Her Ph.D. research involved developing and studying short wavelength (XUV and VUV) lasers and ultrashort pulse laser systems.

She has been a Staff Member at the National Institute of Standards and Technology (NIST), Boulder, CO, since 1998, and has conducted research on high-power ultrashort pulse laser systems, femtosecond pulse characterization, nonlinear pulse propagation in dielectric materials, and high-speed optoelectronic measurements. Prior to working at NIST, she was a Director's Postdoctoral Fellow at Los Alamos National Laboratory, Los Alamos, NM. Her research interests include high-speed optoelectronic and microwave measurements and their calibration and ultrashort pulse laser measurements.